Ciências ULisboa

Instabilities

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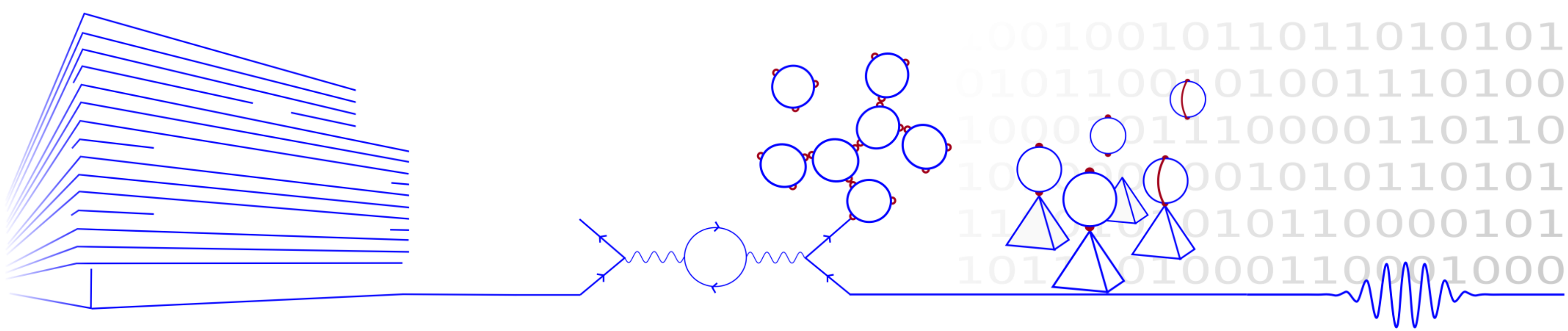
Rodrigo Coelho

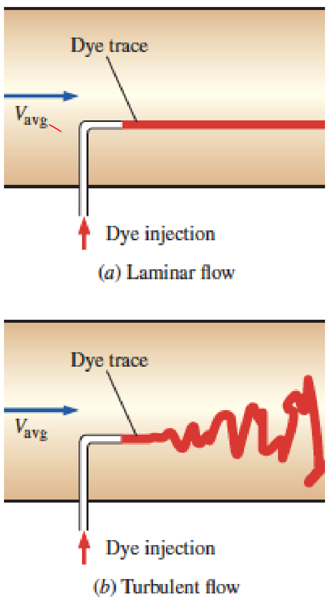
FMC-2022/23

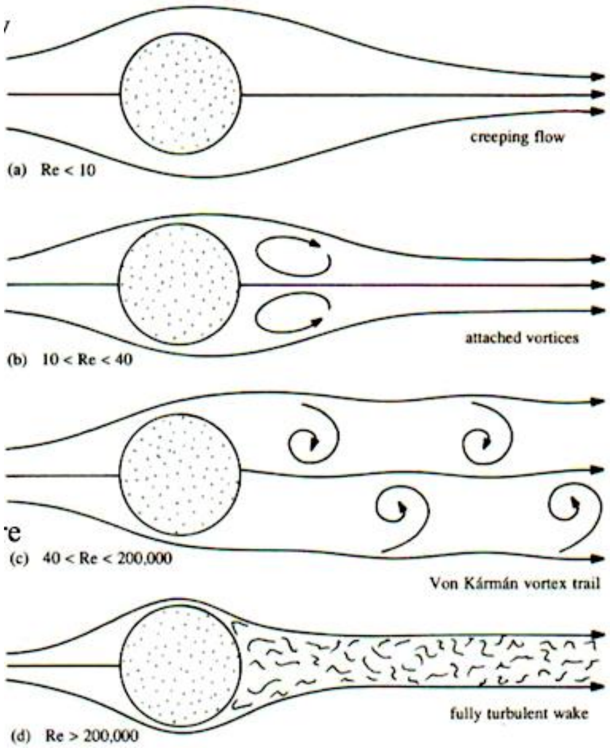


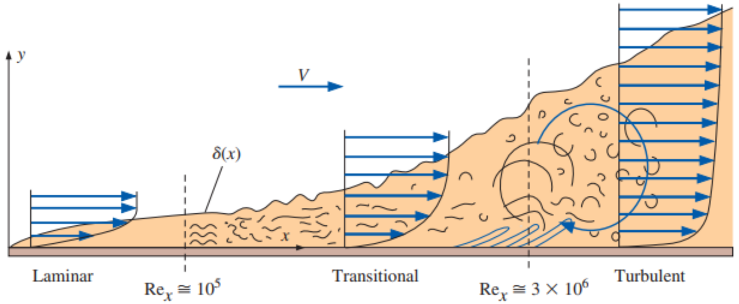
Overview

- Fluid instabilities show up in everyday life, nature and engineering applications. A seemingly stable system may give rise to the development of an instability, which can cascade into turbulence.
- When the system is exposed to a perturbation, some wavelengths will grow, while others will no, governed by the parameters of the flow. This selectivity of specific structure sizes can be determined using linear stability analysis and then accounting for viscosity.
- Once these unstable wavelengths have grown to a substantial degree, the system becomes nonlinear before turbulence eventually sets in.
- Looking at buoyancy-driven instabilities, one can clearly see how certain wavelengths are selected. This can be extended to shear-driven instabilities and to other systems.
- For some flows, simplifications can be made to analyze the specific fluid structures, while for others, only broad conclusions can be drawn about the stability criteria.

Previous chapters

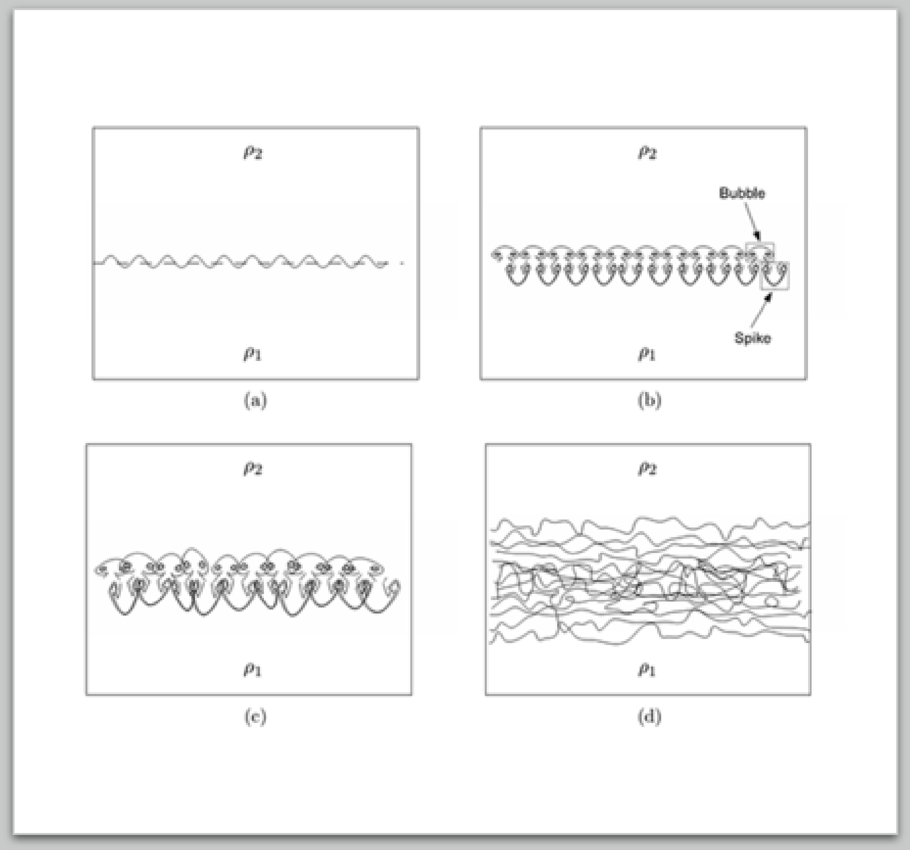


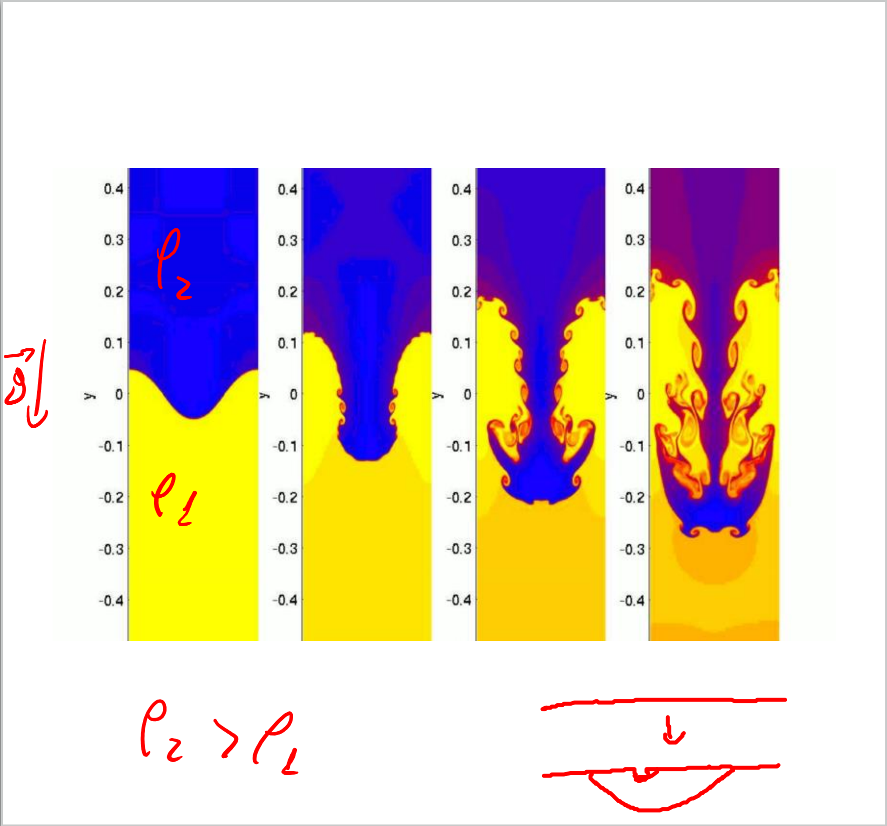




Flow instabilities: https://www.youtube.com/watch?v=8jKZITeUJUQ

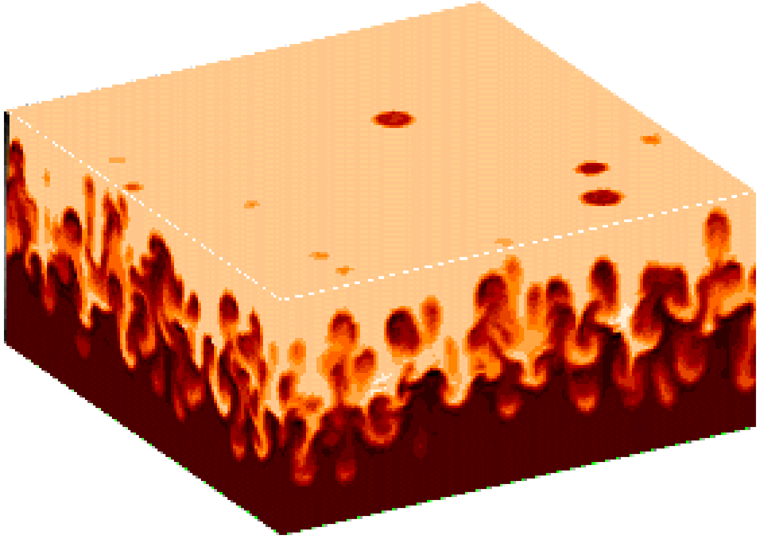
Flow Instabilities | Fluid Mechanics

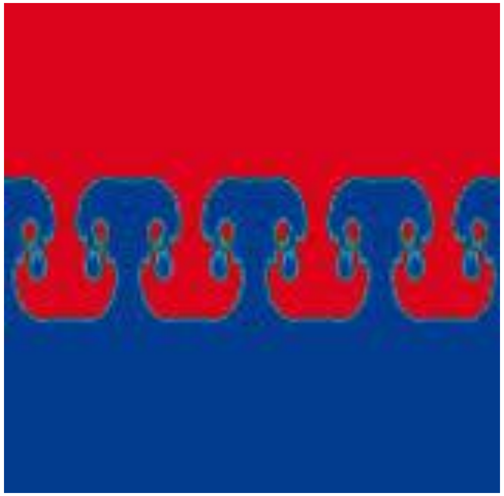


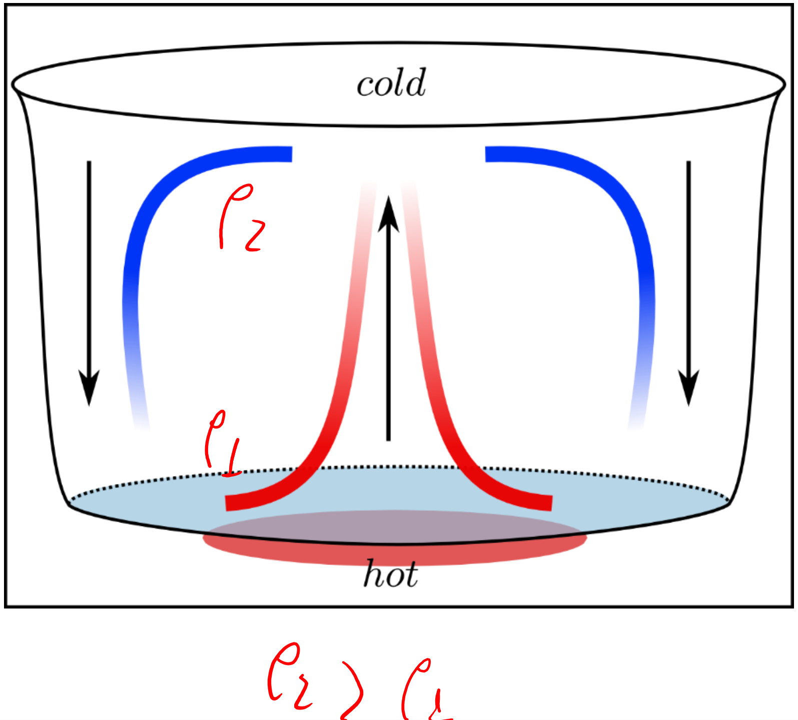


Rayleigh-Taylor

- Instability of an interface between two fluids of different densities, which occurs when the lighter fluid ρ_1 is pushing the heavier fluid ρ_2.

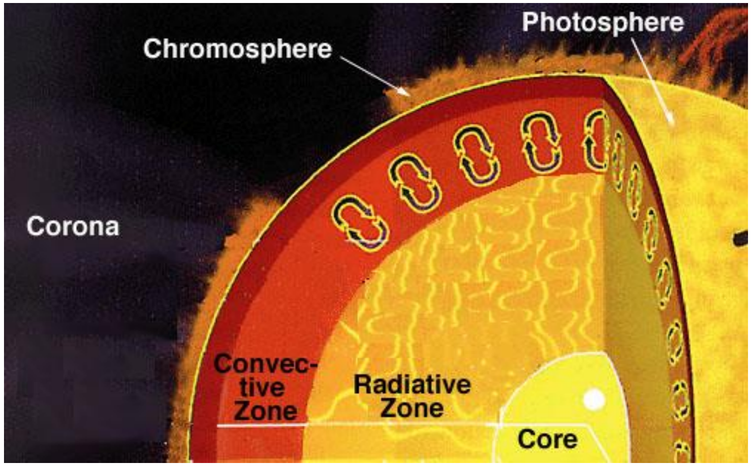


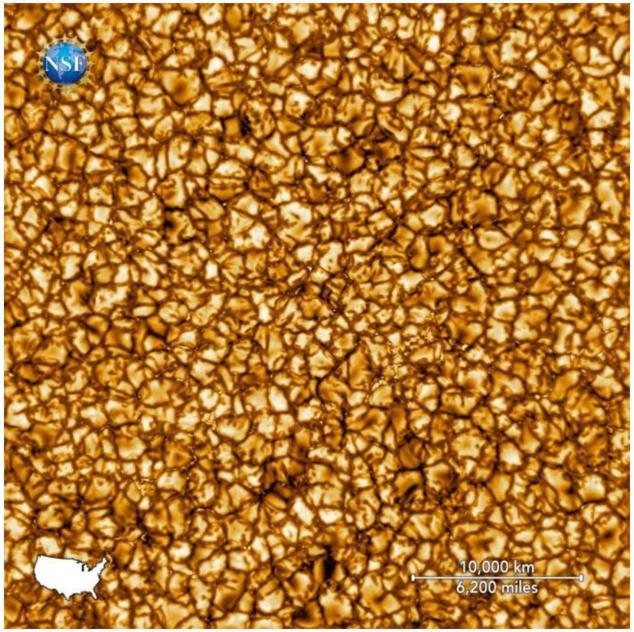


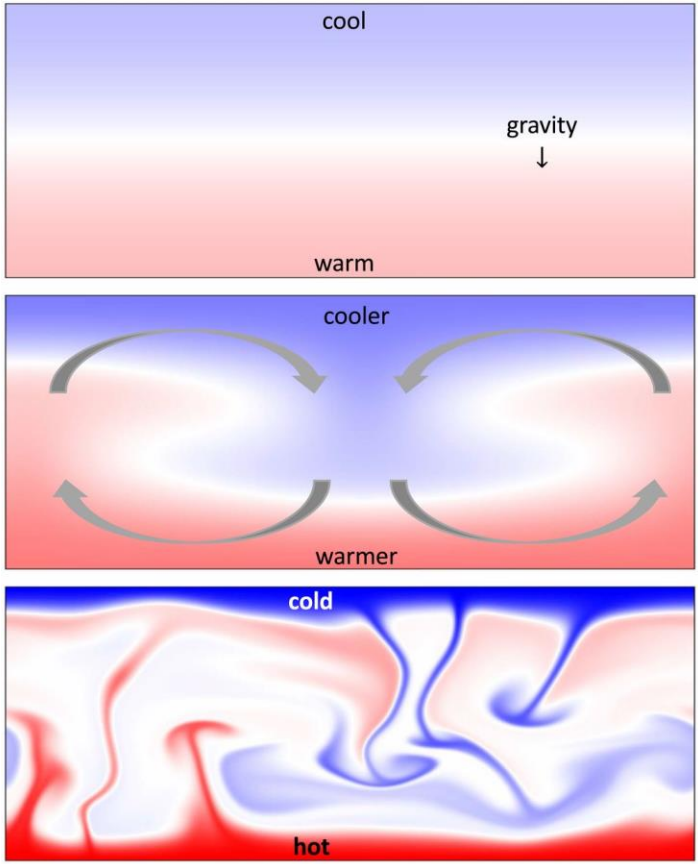


Rayleigh–Bénard

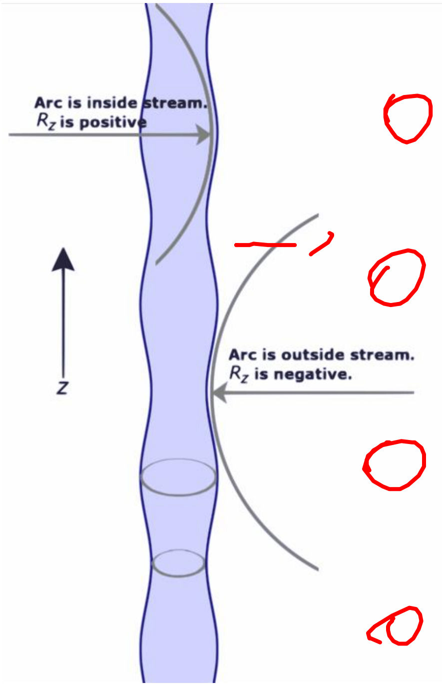
- Instability of a gradient of density of the same fluid (e.g., due to the temperature) under gravity.





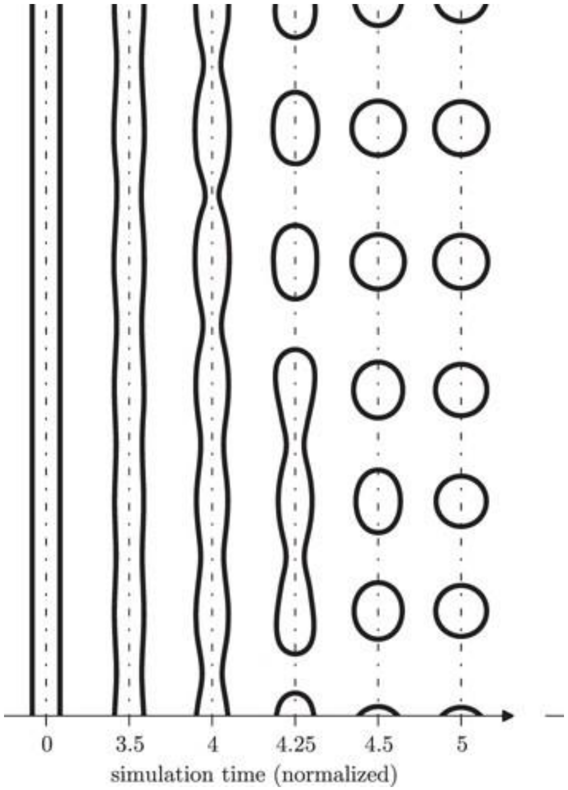


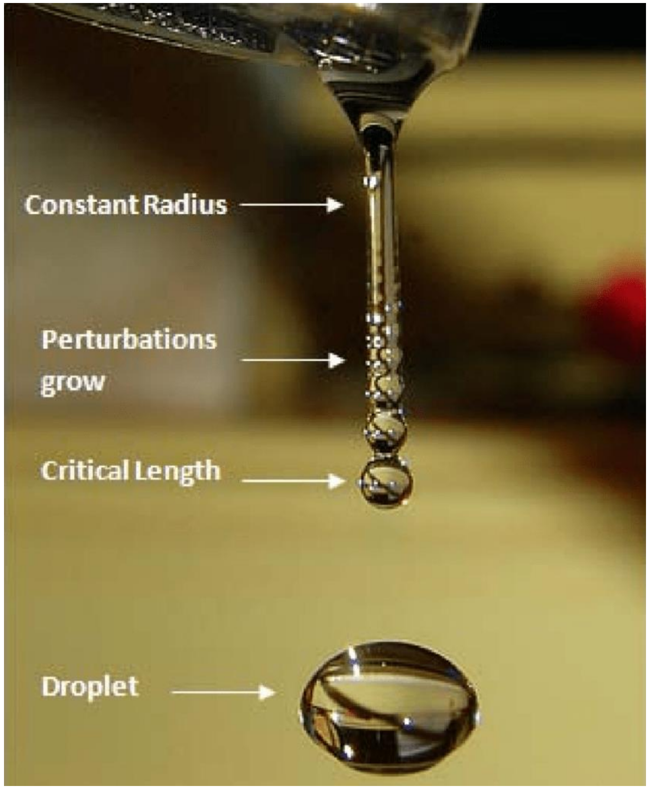
https://blogs.sw.siemens.com/simcenter/blasted-by-the-sun-simcenter-helps-engineer-worlds-largest-solar-telescope/



Rayleigh-Plateau

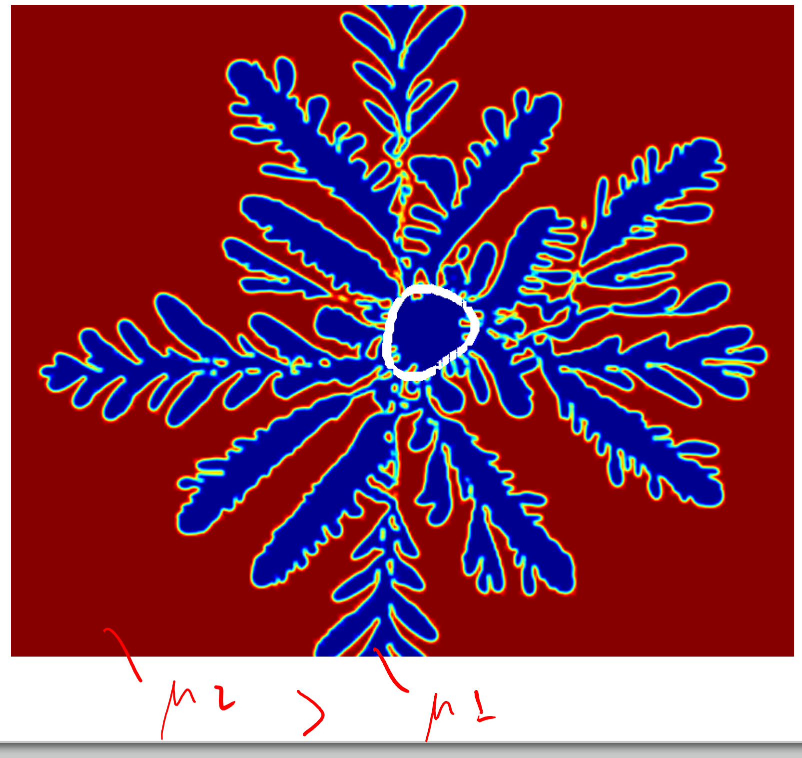
- Instability of a falling stream of fluid that breaks up into smaller packets with the same volume but less surface area.

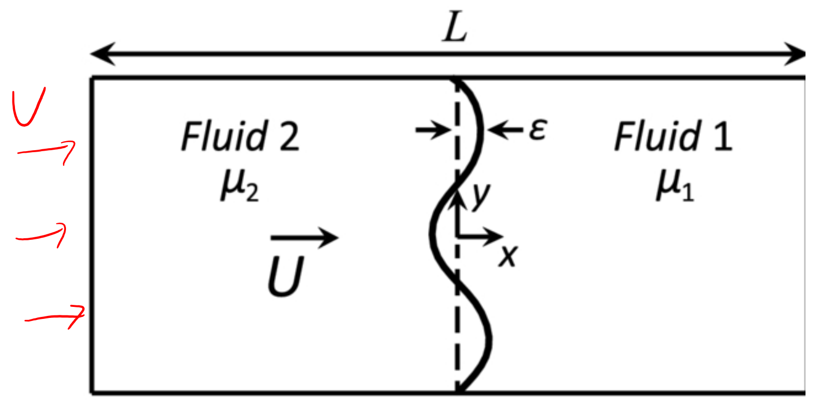




The Plateau–Rayleigh instability is named for Joseph Plateau and Lord Rayleigh. In 1873, Plateau found experimentally that a vertically falling stream of water will break up into drops if its wavelength is greater than about 3.13 to 3.18 times its diameter, which he noted is close to π.[3][4] Later, Rayleigh showed theoretically that a vertically falling column of non-viscous liquid with a circular cross-section should break up into drops if its wavelength exceeded its circumference, which is indeed π times its diameter. (https://en.wikipedia.org/wiki/Plateau%E2%80%93Rayleigh_instability)

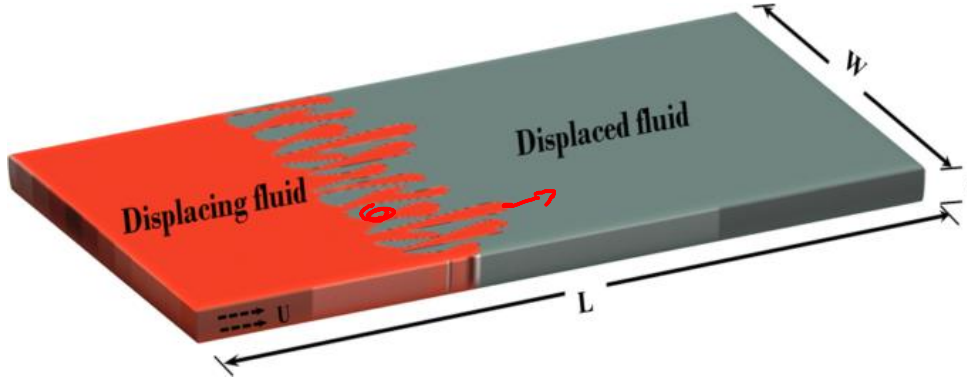
Application: inkjet printing.

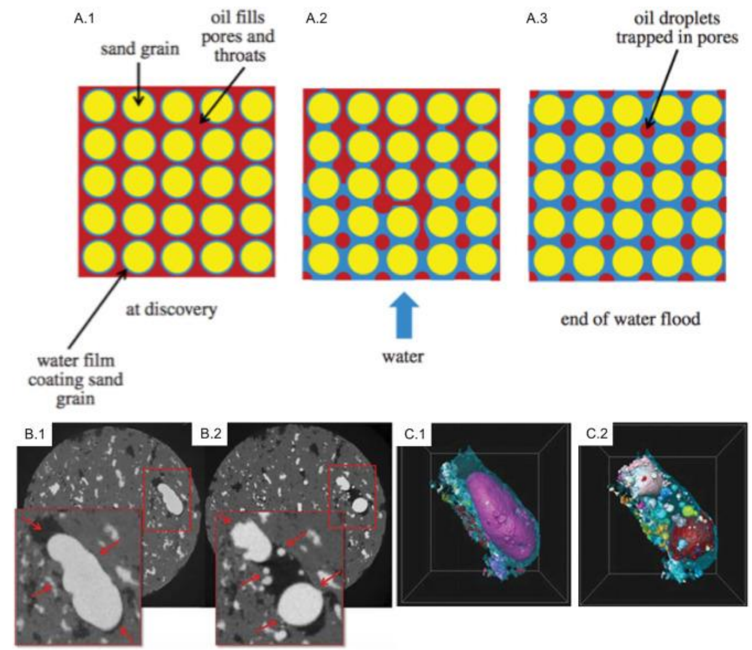




Saffman-Taylor

- Instability that occurs when a more viscous fluid μ_2, is pushed through a less viscous one μ_1 .

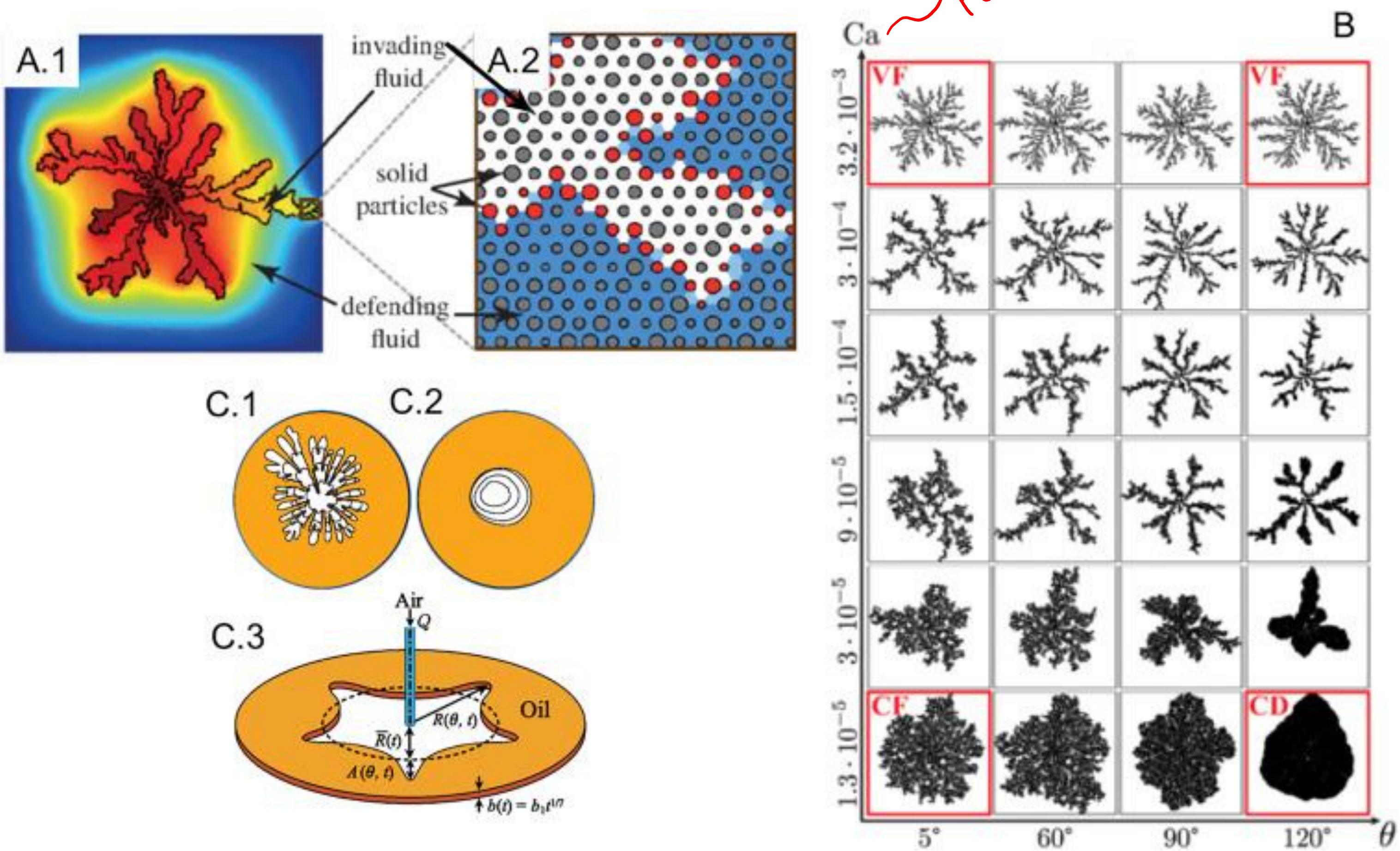




Application: oil extraction

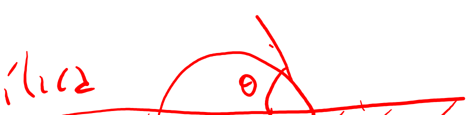
https://www.sciencedirect.com/science/article/pii/S0001868618300174

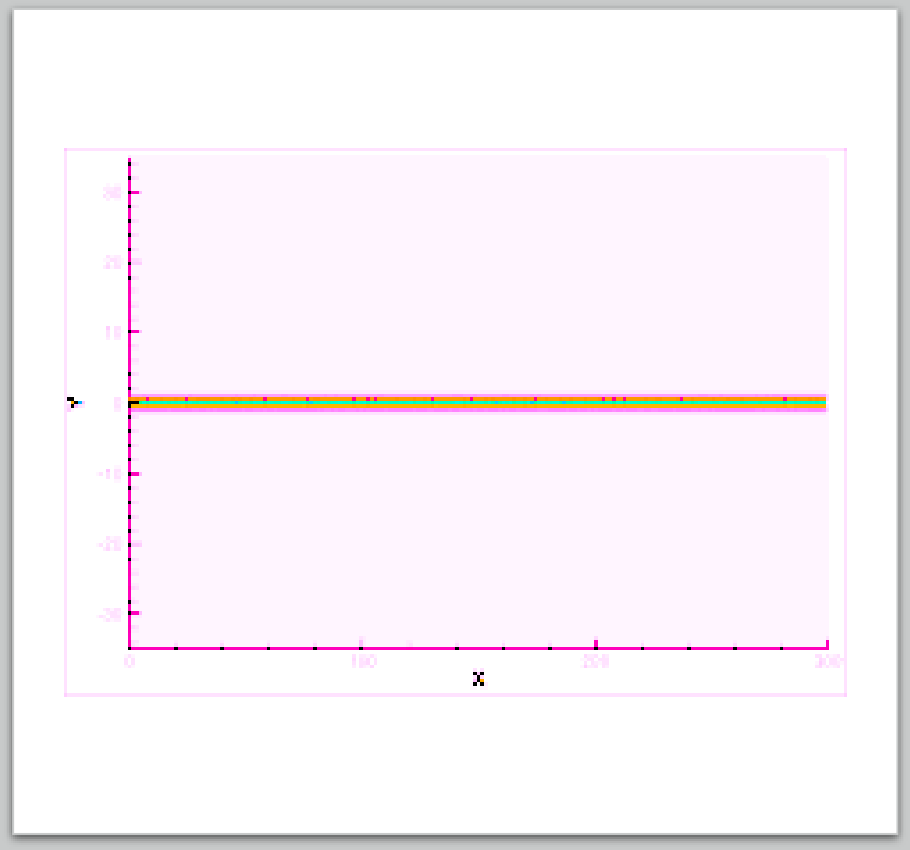
$$\mathrm{Ca} = \frac{\mu V}{\sigma}$$

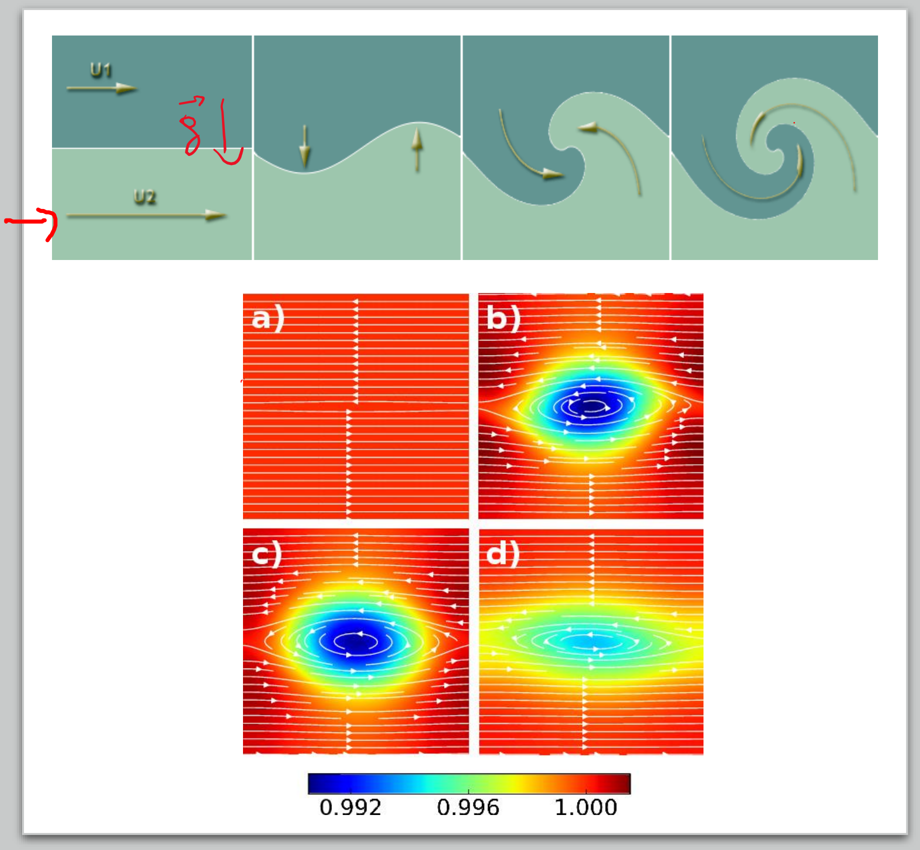


→ https://www.sciencedirect.com/science/article/pii/S0001868618300174

$\theta < 90° \Rightarrow$ Hidrofílica
$\theta > 90° \Rightarrow$ Hidrofóbica

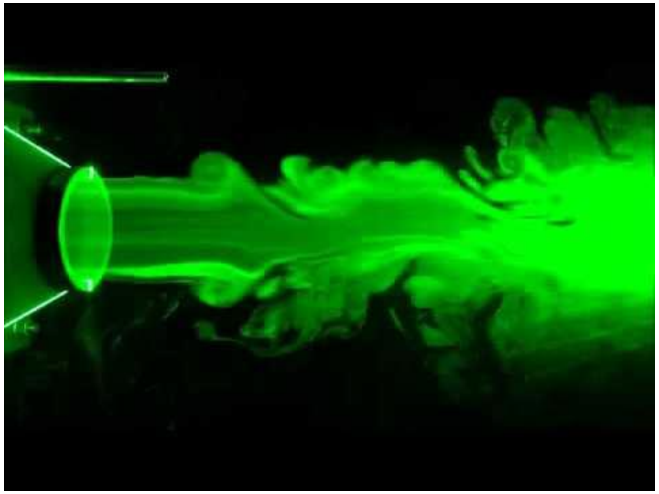






Kelvin-Helmholtz

- Instability that occurs when there is velocity shear in a single continuous fluid, or when there is a velocity difference across the interface between two fluids.



https://www.youtube.com/watch?v=qgamfo86FQo

PHYSICS TODAY

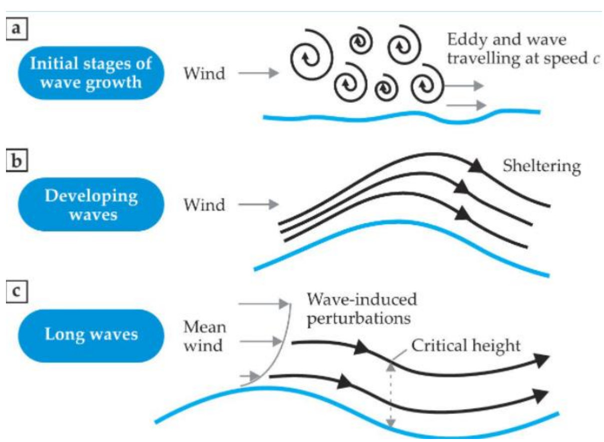
How does the wind generate waves?
November 2021
Physics Today

View

Articles

How does the wind generate waves?

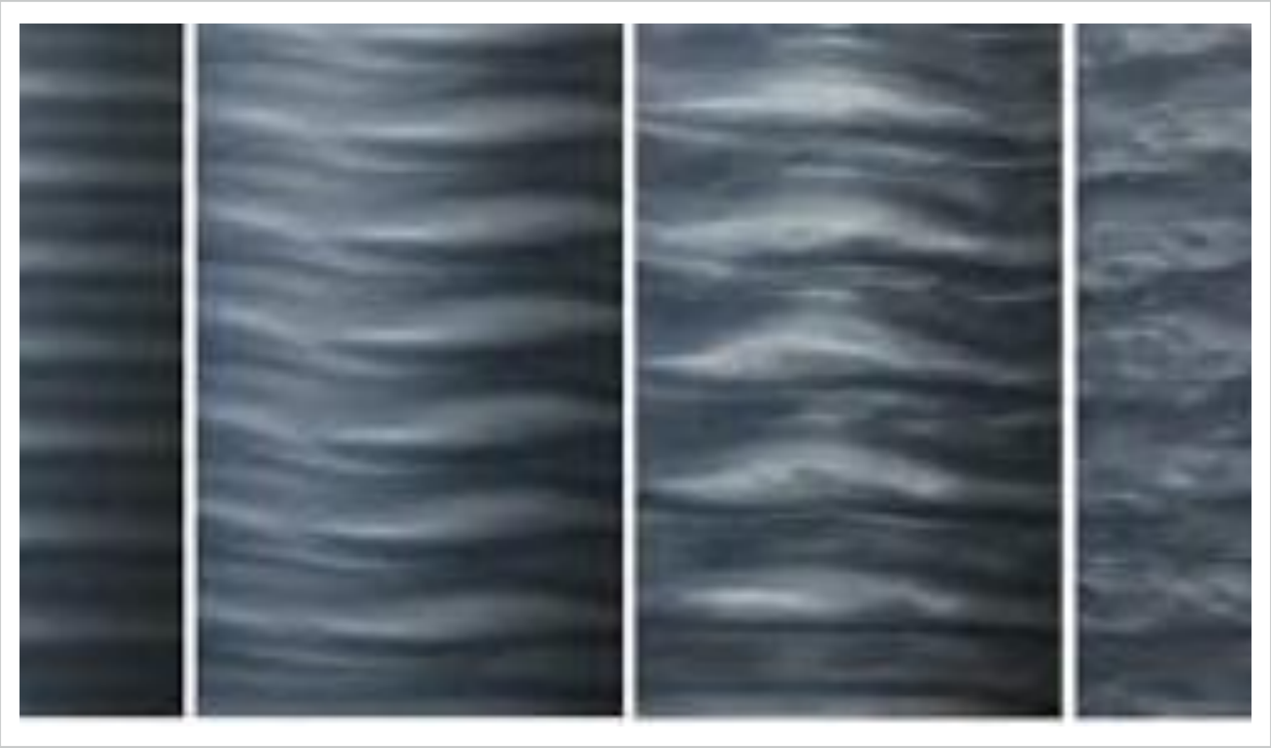
Although the question is a classical problem, the details of how wind transfers energy to waves at the ocean surface remain elusive.



https://magazine.physicstoday.org/2021/10/28/how-does-the-wind-generate-waves/pugpig_index.html

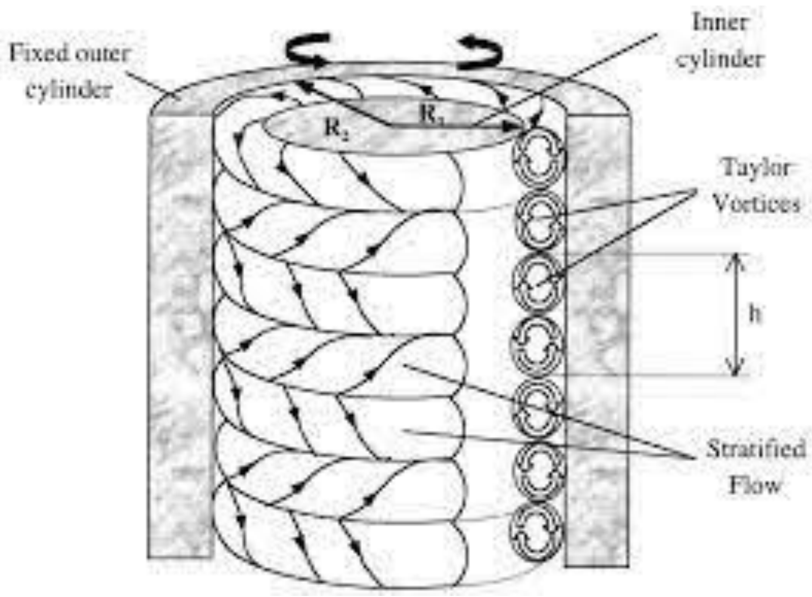
How do ocean currents work? - Jennifer Verduin

https://www.youtube.com/watch?v=p4pWafuvdrY

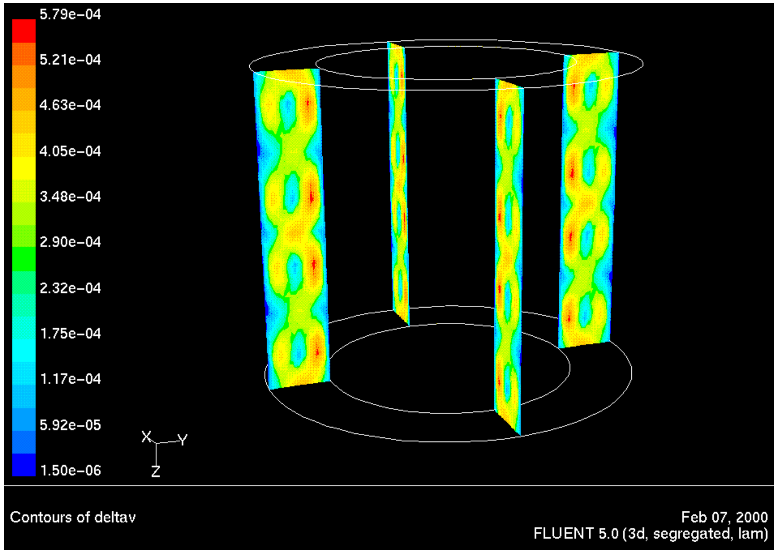


Taylor-Couette

Taylor showed that when the angular velocity of the inner cylinder is increased above a certain threshold, Couette flow becomes unstable and a secondary steady state characterized by axisymmetric toroidal vortices, known as **Taylor vortex** flow, emerges. Subsequently, upon increasing the angular speed of the cylinder the system undergoes a progression of instabilities which lead to states with greater spatio-temporal complexity, with the next state being called **wavy vortex flow.** Beyond a certain Reynolds number there is the onset of turbulence.



Fixed outer cylinder
Inner cylinder
Taylor Vortices
Stratified Flow
h



5.79e-04
5.21e-04
4.63e-04
4.05e-04
3.48e-04
2.90e-04
2.32e-04
1.75e-04
1.17e-04
5.92e-05
1.50e-06
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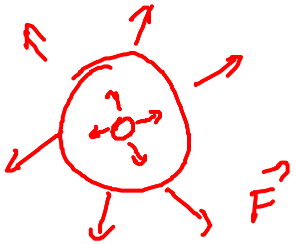
FLUENT 5.0 (3d, segregated, lam)

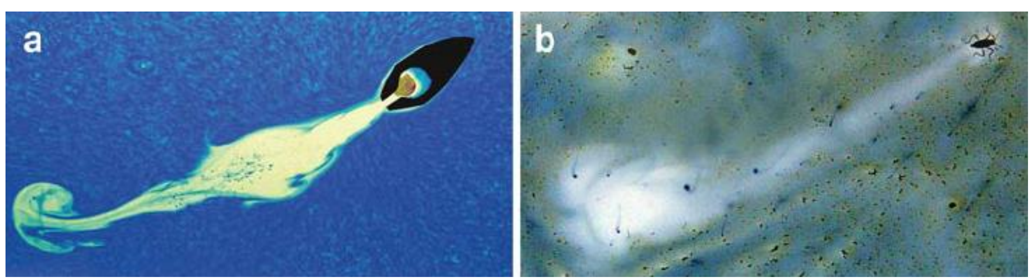


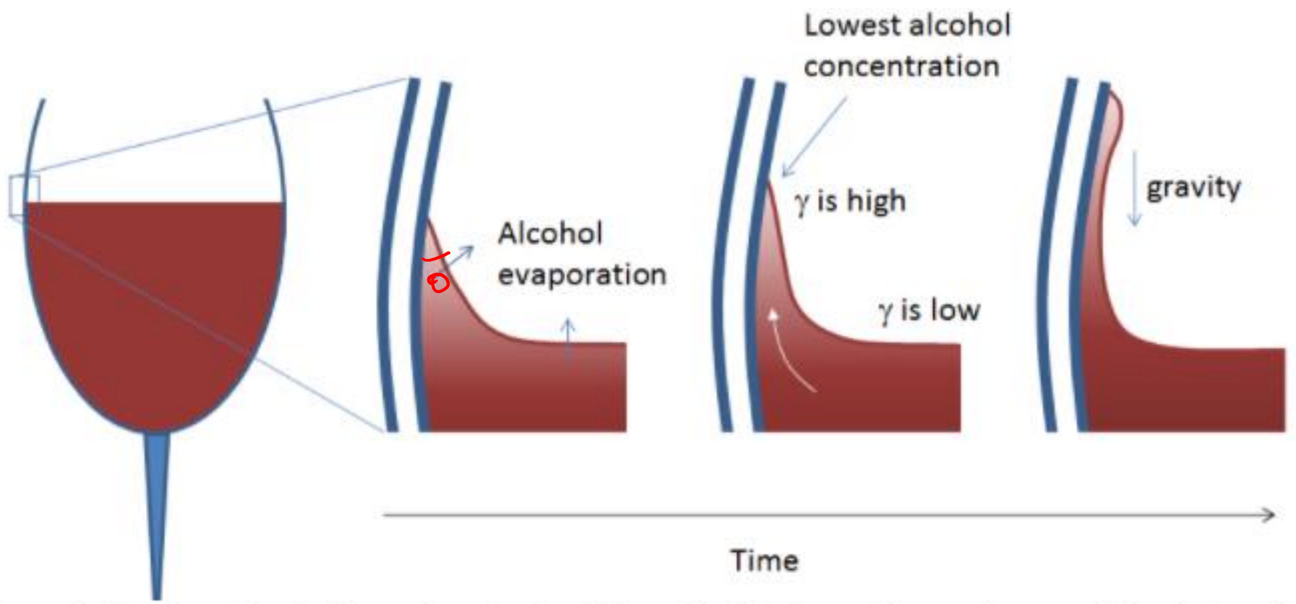
https://en.wikipedia.org/wiki/Marangoni_effect

Since a liquid with a high surface tension pulls more strongly on the surrounding liquid than one with a low surface tension, the presence of a gradient in surface tension will cause the liquid to flow away from regions of low surface tension.

https://www.youtube.com/watch?v=rq55eXGVvis

The Marangoni Effect: How to make a soap propelled boat!

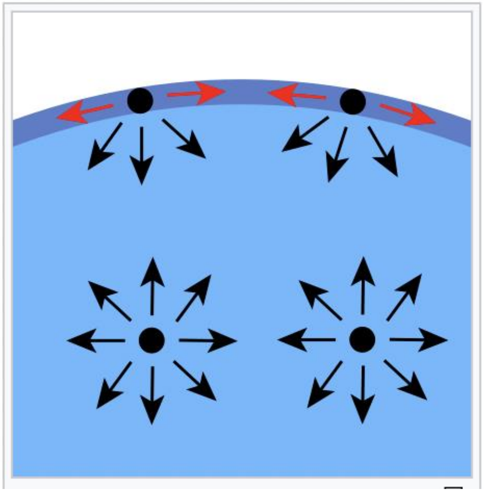




Tears of wine form due to the surface tension (γ) gradient between the meniscus and the flat surface of the wine.

https://br.comsol.com/blogs/tears-of-wine-and-the-marangoni-effect/

Surface tension



$$\gamma = 1\ \frac{\mathrm{dyn}}{\mathrm{cm}} = 1\ \frac{\mathrm{erg}}{\mathrm{cm}^2} = 0,001\ \frac{\mathrm{N}}{\mathrm{m}} = 0,001\ \frac{\mathrm{J}}{\mathrm{m}^2}$$

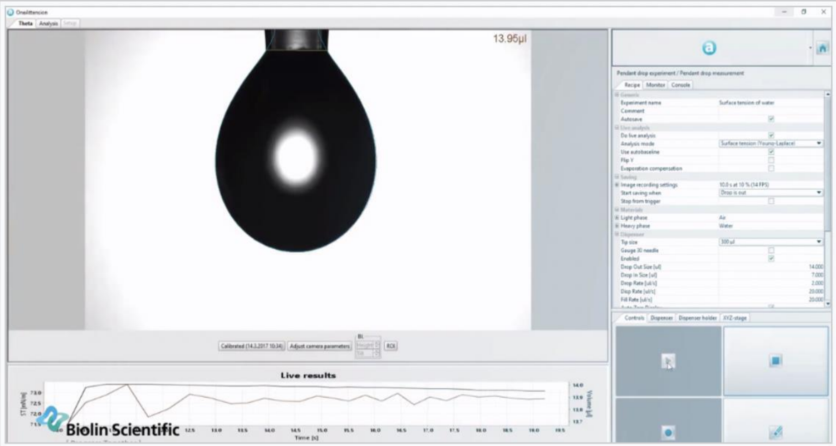
Surface tension is the tendency of liquid surfaces at rest to shrink into the minimum surface area possible. Surface tension is what allows objects with a higher density than water such as razor blades and insects (e.g. water striders) to float on a water surface without becoming even partly submerged.

https://en.wikipedia.org/wiki/Surface_tension

Surface tension

Young–Laplace equation

$$\Delta p = \rho g h - \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$



Optical tensiometers use the Young-Laplace equation to determine liquid surface tension automatically based on pendant droplet shape.

https://en.wikipedia.org/wiki/Young%E2%80%93Laplace_equation

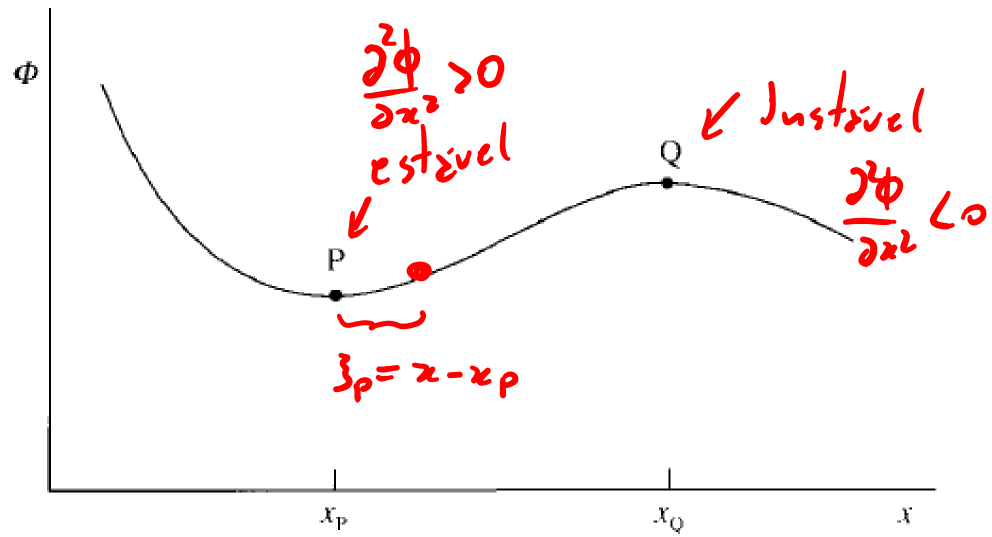


Figure 8.1 Positions of stable (P) and unstable (Q) equilibrium for a part whose potential energy Φ varies with x in the manner shown.

Stability, instability and marginal stability

Faber Chap. 8

As every physicist knows, a dynamical system which is in equilbrium may be *stable* or *unstable*. The simplest case of the distinction is that of a particle of mass m which can move only in one dimension, in circumstances where the particle's potential energy Φ varies with its position x in the manner suggested by fig. 8.1. The particle experiences no force when it is situated at the minimum, P, or at the maximum, Q, and in principle it can remain at rest indefinitely in either of these positions. However, if it is slightly displaced from P it accelerates towards P, whereas if it is slightly displaced from Q it accelerates away from Q; in the first position the particle is stable and in the second it is unstable. Near any minimum such as P the restoring force $\partial\Phi/\partial x$ can normally be expanded as a Taylor series in powers of displacement $\xi_P = x - x_P$. Since it is zero at P itself, an adequate approximation for small values of ξ_P is

$F = +ma = +m\frac{\partial^2 \xi_p}{\partial t^2}$ (1)

$$\frac{\partial \Phi}{\partial x} \approx \left(\frac{\partial^2 \Phi}{\partial x^2}\right)_P \xi_P,$$

in which case the equation of motion of the particle is linear in ξ_P,

$\vec{F} = -\nabla\phi$

$F = -\frac{\partial \phi}{\partial x} \approx -\left(\frac{\partial^2 \phi}{\partial x^2}\right)_p \xi_p$ (2)

(1) = (2) $\Rightarrow$

$$m\frac{\partial^2 \xi_P}{\partial t^2} = -\left(\frac{\partial^2 \Phi}{\partial x^2}\right)_P \xi_P.$$

Oscilador harmónico

$\omega^2 = \frac{1}{m}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_p$

The oscillations which it describes are then simple harmonic, with angular frequency ω_P such that

$$\omega_P^2 = \frac{1}{m}\left(\frac{\partial^2 \Phi}{\partial x^2}\right)_P.$$

An equation of motion similar to (8.1) applies in the neighbourhood of Q, but since $(\partial^2\Phi/\partial x^2)_Q$ is negative the roots for ω are necessarily imaginary, $\omega_Q = \pm i s_Q$ with s_Q real. Hence the displacement $\xi_Q = x - x_Q$ of a particle which starts at rest at $t = 0$ from a position such that $\xi_Q = \xi_o$ is given at later times by

$$\xi_Q \approx \frac{1}{2}\,\xi_o\,(e^{s_Q t} + e^{-s_Q t}),$$

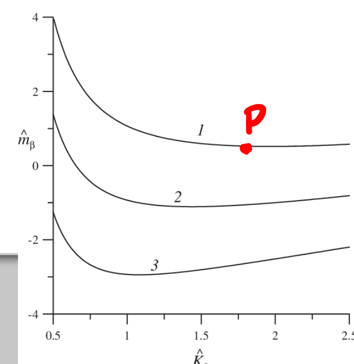
as long as it remains small. If ξ_o is infinitesimal, then by the time the displacement becomes apparent $\exp(s_Q t)$ must be very much greater than unity, in which case $\exp(-s_Q t)$ must be negligible. When a particle leaves a position of unstable equilibrium, therefore, its displacement normally grows in an exponential fashion.

Suppose now that $(\partial\Phi/\partial x)_{\mathrm{P}}$ is necessarily always zero – perhaps because of some symmetry requirement – while $(\partial^2\Phi/\partial x^2)_{\mathrm{P}}$ can be reduced in magnitude and ultimately reversed in sign by altering the external constraints which determine Φ. In that case P is always an equilibrium position, but the equilibrium is stable in one range of the constraints and unstable in an adjacent range. Where the changeover occurs one has

$$\omega_{\mathrm{P}}^2 = \frac{1}{m}\left(\frac{\partial^2\Phi}{\partial x^2}\right)_{\mathrm{P}} = 0,$$

Determinando a condição mínima para a instabilidade.

and this is the condition for what is called *marginal stability*. When it is satisfied, the force experienced by a particle near P is normally determined by $(\partial^3\Phi/\partial x^3)_{\mathrm{P}}$ or, if $(\partial^3\Phi/\partial x^3)_{\mathrm{P}}$ is zero for symmetry reasons, by $(\partial^4\Phi/\partial x^4)_{\mathrm{P}}$; it is then proportional to ξ_{P}^2 or ξ_{P}^3 rather than to ξ_{P}.



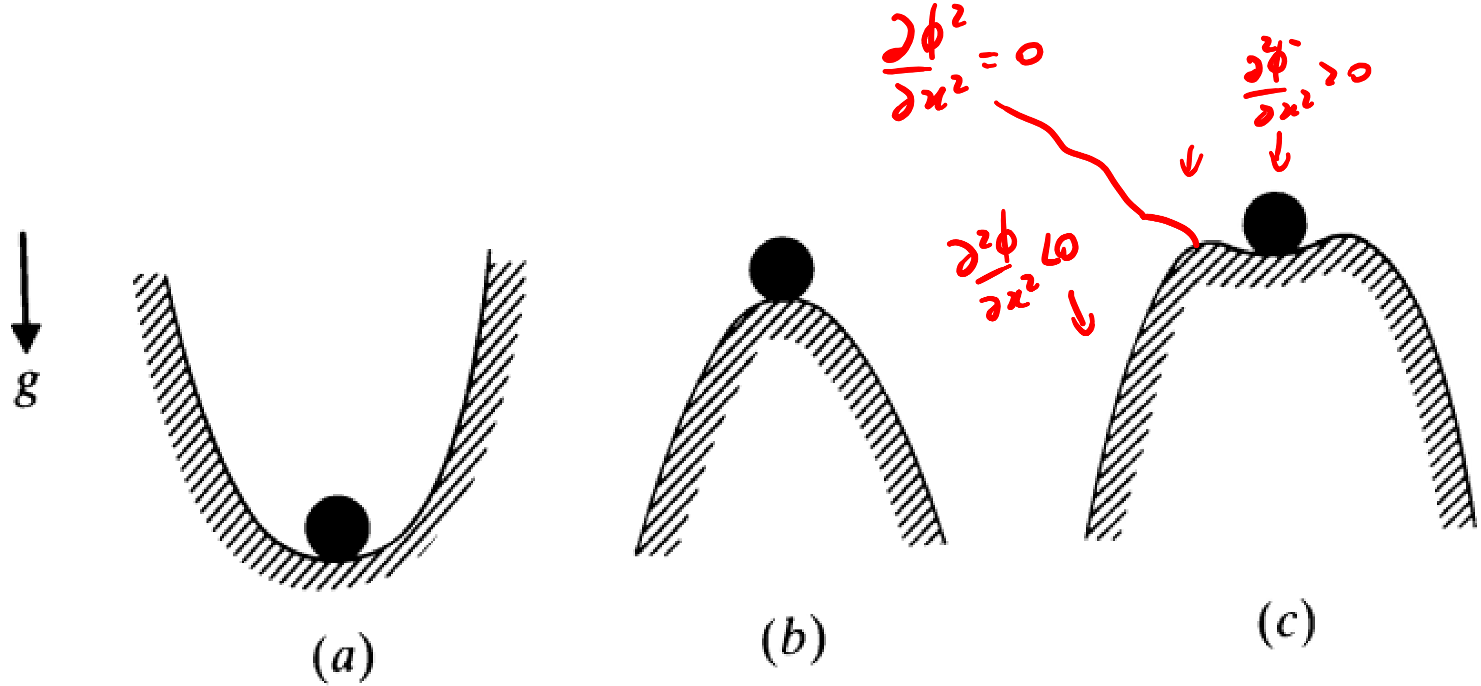


Fig. 9.3. (*a*) A stable state. (*b*) An unstable state. (*c*) A state which is stable to infinitesimal disturbances but unstable to disturbances which exceed some small threshold amplitude.

Dissipative systems

In so far as the above remarks apply to conservative systems they may seem to have little relevance to viscous fluids, which are inherently dissipative. If, however, a particle moving in the potential of fig. 8.1 is subject to a dissipative retarding force proportional to its velocity, the principal effect of this is merely to damp – and perhaps overdamp – oscillations in ξ_P and to slow down the exponential rate of growth of ξ_Q. That does not invalidate the conclusion that P and Q represent states of stable and unstable equilibrium respectively. Indeed, the fluctuations which always accompany dissipation in thermal equilibrium now make it impossible in principle, as well as in practice, for a particle to remain indefinitely at Q. Nor does the existence of dissipation invalidate the conclusion that when, as a result of a continuous change in the form of $\Phi\{x\}$, the equilibrium at P changes from being stable to being unstable, this equilibrium passes through a state of marginal stability.

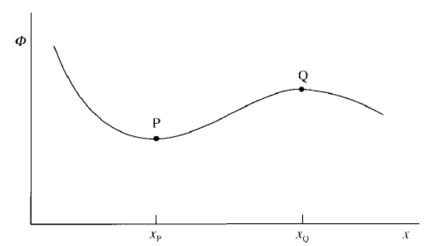


Figure 8.1 Positions of stable (P) and unstable (Q) equilibrium for a par[t]
whose potential energy Φ varies with x in the manner shown.

The general procedure for investigating the stability or otherwise of patterns of fluid flow involves perturbing the pattern in various ways and calculating whether the amplitude – say $\zeta_n\{t\}$ – of each perturbation mode decreases or increases with time; the amplitude may well describe a velocity rather than a displacement, but that is a rather trivial distinction in this context. The modes must be consistent with the boundary conditions to which the fluid is subject, and they should form, like the periodic normal modes of the Euler strut, a complete set in terms of which any possible perturbation may be expanded. The exact equations of motion of the fluid are always non-linear in ζ_n, and one cannot achieve a detailed understanding of what happens once an instability has developed without taking non-linear terms into account. As a first step, however, it may suffice to establish the condition for a state of marginal stability to exist; having done that, one may confidently assert that true stability lies on one side of this condition and instability on the other.

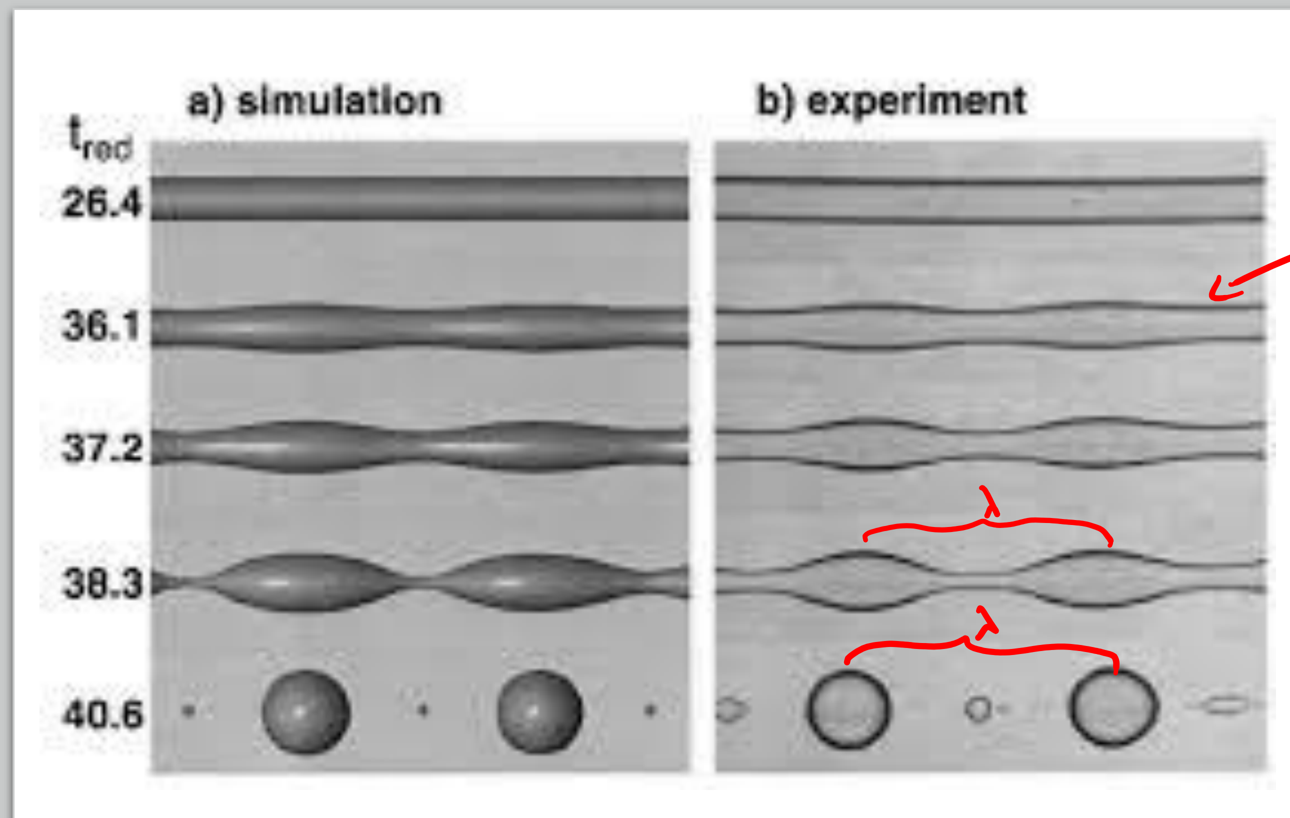
Since marginal stability requires

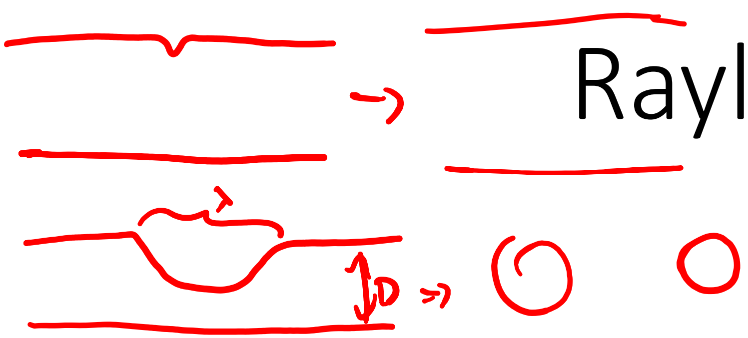
$$s_n = \frac{1}{\zeta_n}\frac{\partial \zeta_n}{\partial t}$$

modo de crescimento mais rápido

$$\frac{\partial \zeta_n}{\partial t} = 0 \tag{8.2}$$

to first order only in ζ_n, the condition for its existence may be established using approximate equations of motion from which all terms which are non-linear in ζ_n have been deleted. If, as is often the case, there are several competing modes of instability, the first to develop once the condition for marginal stability has been exceeded is normally the one for which $s_n\ (= \zeta_n^{-1}\partial\zeta_n/\partial t)$ is largest. Linearised equations of motion suffice to settle this question as well.





Rayleigh-Plateau

$$\sigma = \frac{\text{Energia}}{\text{Área}}$$

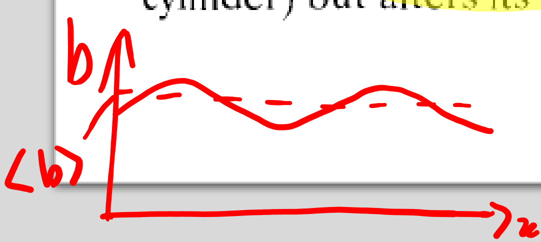
Rayleigh-Plateau

A free jet of water, emerging from a circular orifice, is liable to break up into a regular succession of drops, and according to Plateau's analysis of some observations by Savart the drops are separated by a distance λ which is about 8·8 times the radius a of the jet before it disintegrates. If a stationary cylinder of water could be obtained it would break up in the same way, and indeed the droplets of water which are to be seen on spiders' webs after a damp cold night are probably formed by accretion from layers of dew which are cylindrical when first deposited. The explanation lies in the fact that, volume for volume, spheres have smaller surface areas than cylinders.

Suppose an initially uniform cylinder of liquid to be subject to a small *varicose deformation*, which preserves rotational symmetry about the x axis (the axis of the cylinder) but alters its radius in a periodic fashion from a to

$$b = \langle b \rangle + \zeta_k \cos kx \quad (\zeta_k \ll a). \qquad (1)$$

$$k = \frac{2\pi}{\lambda}$$



The volume of the cylinder per unit length, averaged over an integral number of wavelengths, is

Área de secção média

$$V = \langle \pi b^2 \rangle = \pi \langle b \rangle^2 + \frac{1}{2} \pi \zeta_k^2, \quad (2)$$

$= \pi a^2$ (Eq.(1))

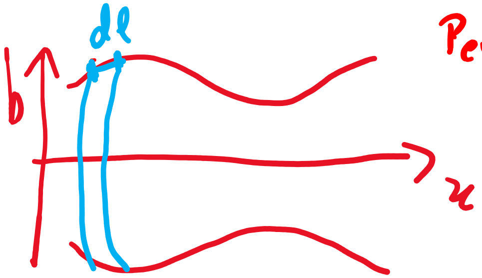
$$\langle \dots \rangle = \frac{\int_0^L (\dots)\, dx}{L}$$

and since this must equal the initial volume per unit length, πa^2, we have

$$\rightarrow \langle b \rangle = \sqrt{a^2 - \tfrac{1}{2}\zeta_k^2} \approx a - \frac{\zeta_k^2}{4a}. \quad (3)$$

$$\sqrt{1+x} \simeq 1 + \frac{x}{2}$$

Thus the surface area of the cylinder per unit length, similarly averaged, is



Perímetro médio

$$A = \left\langle 2\pi b \sqrt{1 + \left(\frac{\mathrm{d}b}{\mathrm{d}x}\right)^2} \right\rangle$$

(Eq.(1)) (Eq.(1))

$$\approx 2\pi a + \frac{\pi \zeta_k^2}{2a} \left\{ (ka)^2 - 1 \right\}.$$

(Eq.(3))

$$dl = \sqrt{dx^2 + db^2} = dx \sqrt{1 + \left(\frac{db}{dx}\right)^2}$$

$$\phi = \sigma \cdot A$$

$$\frac{\partial^2 \phi}{\partial \zeta_k^2} = 0 = \frac{\partial^2 A}{\partial \zeta_k^2}$$

In this problem there is no gravitational term to consider, and it is the surface free energy per unit length, σA, which plays the role of the potential energy Φ of §8.1. The condition for marginal stability is $\partial^2 A/\partial\zeta_k^2 = 0$, equivalent to

$$\boxed{k = k_c = \frac{1}{a}.}$$

$\lambda_c = 2\pi a \simeq 6{,}2\, a$

Se $\lambda > \lambda_c$, há instabilidade.

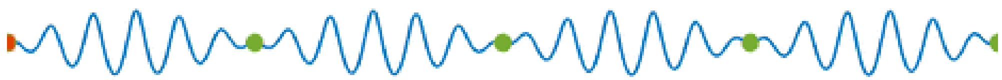
The cylinder is inherently unstable, as Plateau was the first to note, to any periodic deformation for which k is less than k_c, i.e. for which the wavelength λ is greater than $2\pi a$.

$\frac{1}{\zeta_k}\frac{\partial \zeta_k}{\partial t}$

To find the rate of growth of a mode for which $k < k_c$ one may follow the routine procedure outlined in §8.2. Provided that the viscosity of the liquid may be neglected, i.e. provided that potential theory may be employed, it is not difficult to calculate the fluid velocity $\boldsymbol{u}\{x, r\}$ associated with rate of change of ζ_k. It is described by a flow potential ϕ which is a solution of Laplace's equation proportional to $\cos(kx) f\{r\}(\partial\zeta_k/\partial t)$; the function $f\{r\}$ involves Bessel functions. Hence the constant of proportionality relating the fluid's mean kinetic energy per unit length to $(\partial\zeta_k/\partial t)^2$ may be found, and the equation of motion relating $\partial^2\zeta_k/\partial t^2$ to ζ_k follows immediately. According to Rayleigh, s_k, which is zero where $k = k_c$, reaches a maximum where $k = 0{\cdot}697 k_c$ or where $\boxed{\lambda = 9{\cdot}02a}$, in reasonable agreement with Savart's observations. The 2% discrepancy, in the wrong direction to be due to viscosity, is attributable to experimental error.

modo que cresce mais rápido.

A dispersion relation relates the wavenumber of a wave to its frequency



Frequency dispersion of surface gravity waves on deep water. The ■ red square moves with the phase velocity, and the ● green dots propagate with the group velocity. In this deep-water case, the phase velocity is twice the group velocity. The ■ red square traverses the figure in the time it takes the ● green dot to traverse half.

Deep water waves [edit]

Further information: Dispersion (water waves) and Airy wave theory

The dispersion relation for deep water waves is often written as

$$\omega = \sqrt{gk},$$

where g is the acceleration due to gravity. Deep water, in this respect, is commonly denoted as the case where the water depth is larger than half the wavelength.[4] In this case the phase velocity is

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}},$$

and the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{1}{2} v_p.$$

Waves on a string [edit]

Further information: Vibrating string

For an ideal string, the dispersion relation can be written as

$$\omega = k\sqrt{\frac{T}{\mu}},$$

where T is the tension force in the string, and μ is the string's mass per unit length. As for the case of electromagnetic waves in vacuum

Electromagnetic waves in a vacuum [edit]

For electromagnetic waves in vacuum, the angular frequency is proportional to the wavenumber:

$$\omega = ck.$$

This is a *linear* dispersion relation. In this case, the phase velocity and the group velocity are the same:

$$v = \frac{\omega}{k} = \frac{d\omega}{dk} = c;$$

they are given by c, the speed of light in vacuum, a frequency-independent constant.

$$v = \lambda . f = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

$$\Rightarrow \boxed{\omega = v k}$$

(Source: https://en.wikipedia.org/wiki/Dispersion_relation)

Rayleigh-Taylor instability

The Rayleigh–Taylor instability arises when a vessel which contains two fluids separated by a horizontal interface – one at least of the fluids must of course be a liquid – is suddenly inverted so that the heavier fluid lies above the lighter one. The gravitational potential energy of the system, which was at its minimum value before inversion, is now at its maximum, and although the system is still in equilibrium while the interface remains horizontal the equilibrium is clearly liable to be unstable. Whether or not it is actually unstable with respect to any particular perturbation depends upon whether the gravitational energy which this releases is greater or less than the increase in surface free energy. The system is marginally stable with respect to the perturbation when the two are equal.

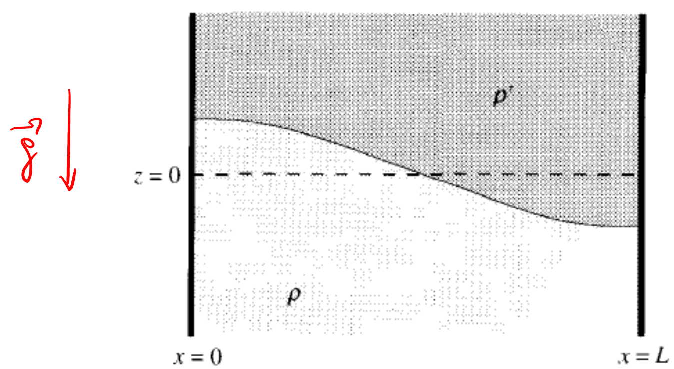
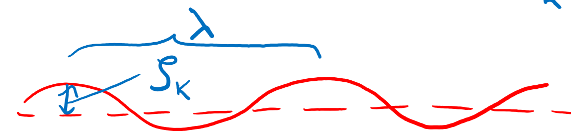


Figure 8.3 A layer of one fluid with a denser fluid above it, in a container of width L, is stabilised by surface tension against the perturbation suggested here provided that (8.6) is satisfied.

Further reading: Chandrasekhar, S. (2013). Hydrodynamic and hydromagnetic stability. Courier Corporation.

All possible small perturbations of the surface may be expressed in terms of their Fourier components, a typical Fourier component involving a vertical displacement of the interface



$$\zeta = \zeta_k\{t\}e^{ik\cdot r},$$

where r is a vector which lies in the $z = 0$ plane, i.e. the plane of the undisturbed interface. Per unit area of the interface, the reduction in gravitational potential energy associated with a single wave of this form, averaged over any integral number of wavelengths, is [(5.29)]

$$\frac{1}{4}(\rho' - \rho)g\zeta_k^2,$$

where ρ' and ρ are the densities of the heavier and lighter fluids respectively. The increase of the surface free energy, similarly averaged, is

$$\sigma\left\langle\left\{1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right\}^{1/2} - 1\right\rangle \approx \frac{1}{4}\sigma k^2\zeta_k^2$$

to second order in ζ_k, where σ is the interfacial surface tension. Marginal stability is therefore only possible for one wavevector k_c, such that

$$(\rho' - \rho)g = \sigma k_c^2. \qquad (8.4)$$

In order to find the rate at which modes for which $k < k_c$ grow in amplitude, one needs to know how the velocities of each fluid depend upon $\partial\zeta_k/\partial t$. With that information at one's disposal, one may follow the routine procedure of evaluating the mean kinetic energy per unit area and hence the total energy, a sum of gravitational and surface terms proportional to ζ_k^2 and kinetic terms proportional to $(\partial\zeta_k/\partial t)^2$; by equating the time derivative of the total energy to zero one may then obtain, after cancellation of a factor $\partial\zeta_k/\partial t$, a linear equation of motion relating ζ_k to $\partial^2\zeta_k/\partial t^2$ which provides the required answer. We, however, can make use of a result already available as (5.40), which tells us, in the notation of §8.1, that the dispersion relation for waves on the interface is

angular frequency

$$\omega_k^2 = -\frac{\rho' - \rho}{\rho' + \rho} gk + \frac{\sigma k^3}{\rho' + \rho}, \quad \text{with } s_k = 0,$$

$\omega = \frac{2\pi}{\tau}$

as long as k is greater than the critical wavevector which (8.4) describes, which implies that when $k < k_c$ we have

$$s_k^2 = \frac{\rho' - \rho}{\rho' + \rho} gk - \frac{\sigma k^3}{\rho' + \rho}, \quad \text{with } \omega_k = 0.$$

$\exp\{-i(\omega_n + is_n)t\}$

The value of k, say $k_{\max}$, which maximises s_k and hence the rate of growth is clearly such that

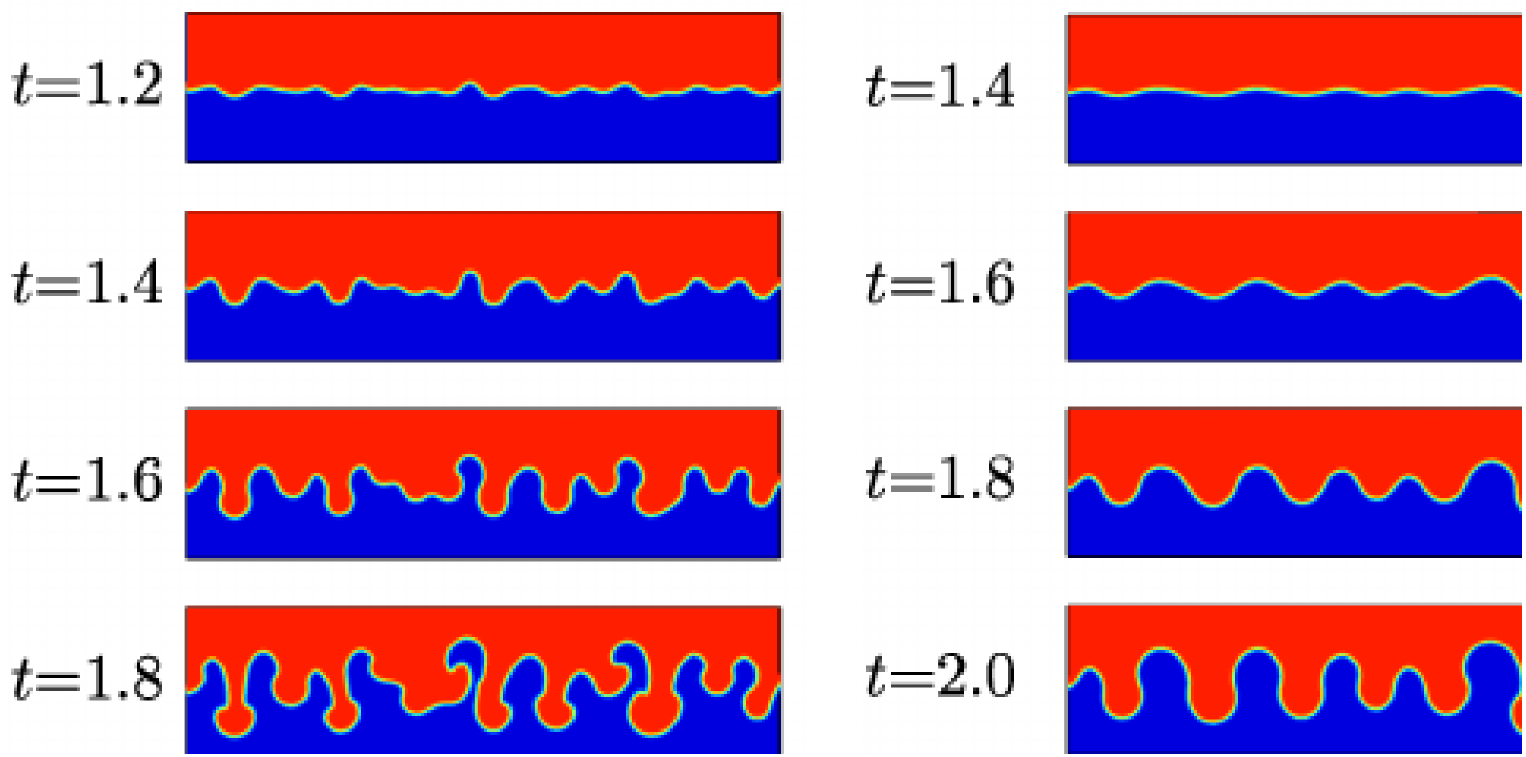
$$(\rho' - \rho)g = 3\sigma k_{\max}^2,$$

so

$$k_{\max} = \frac{k_c}{\sqrt{3}}. \tag{8.5}$$

We may infer from the above results that if it were possible to invert almost instantaneously a *large* vessel containing two fluids, so large that the boundary conditions imposed virtually no limitations on the allowed values of $\boldsymbol{k}$, the contents would be inherently unstable. The interface would inevitably develop corrugations whose periodicity would be the wavelength associated with $k_{\max}$, i.e. $2\pi\sqrt{3\sigma/(\rho' - \rho)g}$, which amounts to about 3 cm when the heavier fluid is water and the lighter one is air. In practice, however, rapid inversion is possible only with small vessels, and the fact that liquid inside an inverted bottle is stabilised by surface tension if the opening of the bottle is small enough must be familiar to every reader. For simplicity, suppose the vessel to be a rectangular one, with vertical sides and a cross-section in the $z = 0$ plane of which the larger dimension is L. The smallest non-zero value of k consistent with the boundary conditions [fig. 8.3 and some remarks about the boundary conditions applicable to water waves at the start of §5.8] is then π/L. In that case the inverted contents are stable provided that $\pi/L > k_c$, i.e. provided that

$$L < \pi\sqrt{\frac{\sigma}{(\rho' - \rho)g}}. \tag{8.6}$$



Rayleigh-Taylor

Saffman-Taylor instability

The Saffman–Taylor instability arises, or may arise, when two fluids of different viscosity are pushed by a pressure gradient through a Hele Shaw cell [§6.8] or allowed to drain through such a cell under their own weight. It would be of little practical importance were it not for the fact that creeping flow in a Hele Shaw cell is the two-dimensional analogue of creeping flow through a porous medium [§6.13]. Something very like the Saffman–Taylor instability frustrates attempts to extract, by pushing it out with pressurised water, the last traces of oil from oil wells. Theoretically, the instability has features in common with the Rayleigh–Taylor instability discussed in §8.2; it differs in that the equibrium state is a dynamic one, in which the interface between the two fluids is moving rather than stationary, but the analysis required is nevertheless distinctly similar.

Suppose the cell to be horizontal, in which case the effects of gravity may be ignored. Suppose it to be bounded by straight edges at $y = \pm\frac{1}{2}L$, and suppose there to be pressure gradients which are driving the fluid contents in the $+x$ direction with some uniform velocity U. In the equilibrium state whose stability we are to investigate, the interface between the two fluids is the straight line $x = Ut$. Where $x < Ut$, the viscosity is η'; where $x > Ut$, the viscosity is η. According to (6.47), the pressure gradients needed to maintain this motion are given in the two regions by

$$\frac{\partial p'}{\partial x} = -\frac{12\eta' U}{d^2}, \quad \frac{\partial p}{\partial x} = -\frac{12\eta U}{d^2},$$

where d is the thickness of the cell. The pressures p' and p are not necessarily equal at the interface, because the interface is liable to be curved in the vertical (z) direction. Provided that this curvature is constant, however, it does not affect the results of the analysis, so we may as well ignore it and write

$$p' = -\frac{12\eta' U}{d^2}(x - Ut) + p_o, \quad p = -\frac{12\eta U}{d^2}(x - Ut) + p_o,$$

for the equilibrium state, where p_o does not depend upon x.

Now suppose that the interface is perturbed, in such a way that at time t it lies at $x = X$, where

$$X = Ut + \zeta_k e^{iky}.$$

There must be some corresponding perturbation in p' and p, and it must have the same periodicity in the y direction. However, p' and p obey Laplace's equation in two dimensions [§6.8], so any perturbing term which varies like $\exp(iky)$ must vary like $\exp(\pm kx)$ [(5.12)]. Since the perturbation cannot affect the pressure at large distances from the interface, the perturbed pressures presumably have the form

$$p' = -\frac{12\eta' U}{d^2}(x - Ut) + p_o + A' e^{k(x-Ut)} e^{iky}$$

$$p = -\frac{12\eta U}{d^2}(x - Ut) + p_o + A e^{-k(x-Ut)} e^{iky}$$

when k is positive, where the coefficients A' and A are to be determined by reference to the boundary conditions at the interface.

These boundary conditions, applicable in each case at $x = X$, and linearised by omission of terms which are of higher than first order in A or ζ_k are as follows.

(i) $$\langle u' \rangle_x = \langle u \rangle_x = \frac{\partial X}{\partial t},$$

where $\langle \boldsymbol{u} \rangle$ is the mean velocity described by (6.47), or

$$-\frac{d^2}{12\eta'}\frac{\partial p'}{\partial x} = -\frac{d^2}{12\eta}\frac{\partial p}{\partial x} = U + \frac{\partial \zeta_k}{\partial t}\,\mathrm{e}^{\mathrm{i}ky}.$$

To first order this corresponds to

$$-\frac{d^2 k}{12\eta'}A' = -\frac{d^2 k}{12\eta}A = \frac{\partial \zeta_k}{\partial t}\,\mathrm{e}^{\mathrm{i}ky}. \tag{8.7}$$

(ii) $$p' - p = -\sigma\frac{\partial^2 X}{\partial y^2} = \sigma k^2 \zeta_k\,\mathrm{e}^{\mathrm{i}ky},$$

where σ is the interfacial surface tension. To first order this corresponds to

$$A' - A = \left\{\frac{12U}{d^2}(\eta' - \eta) + \sigma k^2\right\}\zeta_k \exp(\mathrm{i}ky). \tag{8.8}$$

It is a trivial exercise to eliminate A' and A from (8.7) and (8.8), and so to obtain the result

$$s_k = \frac{1}{\zeta_k}\frac{\partial \zeta_k}{\partial t} = \frac{1}{\eta' + \eta}\left\{-U(\eta' - \eta)k - \frac{\sigma d^2 k^3}{12}\right\}. \tag{8.9}$$

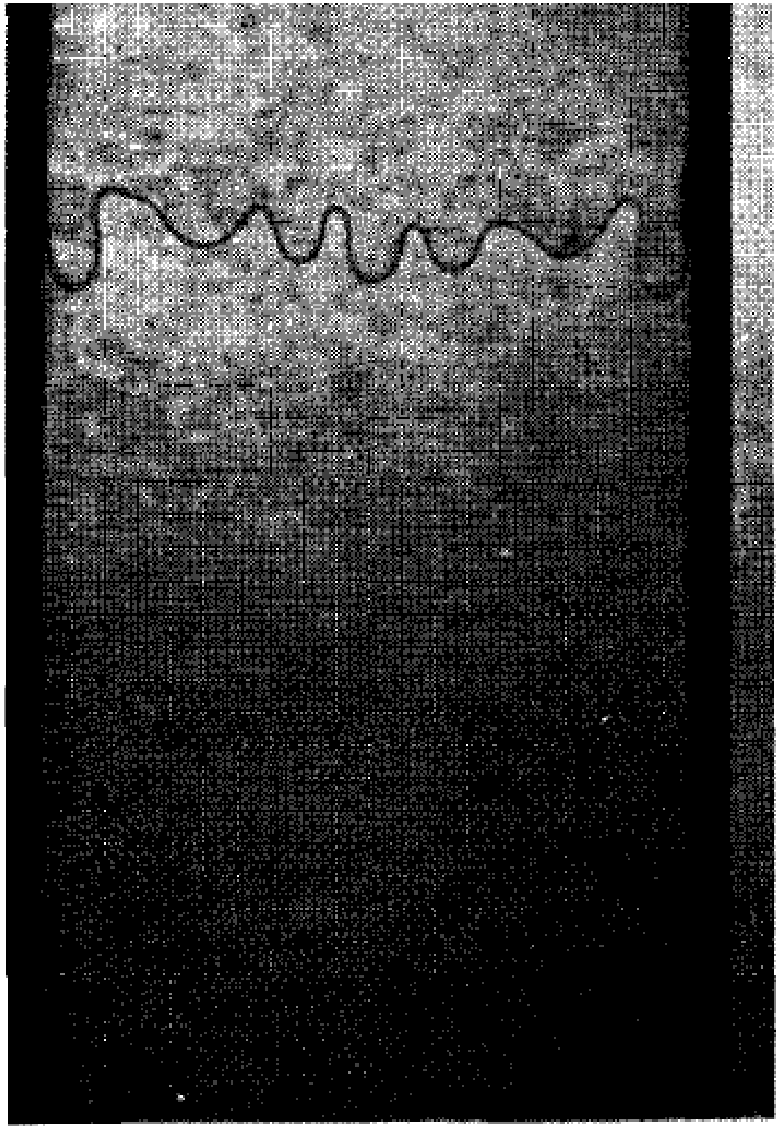
Thus if $\eta < \eta'$ the interface is stable for all k. When $\eta > \eta'$, however, i.e. when a viscous fluid is being displaced by a less viscous one, it is marginally stable with respect to a perturbation for which $k = k_c$, where

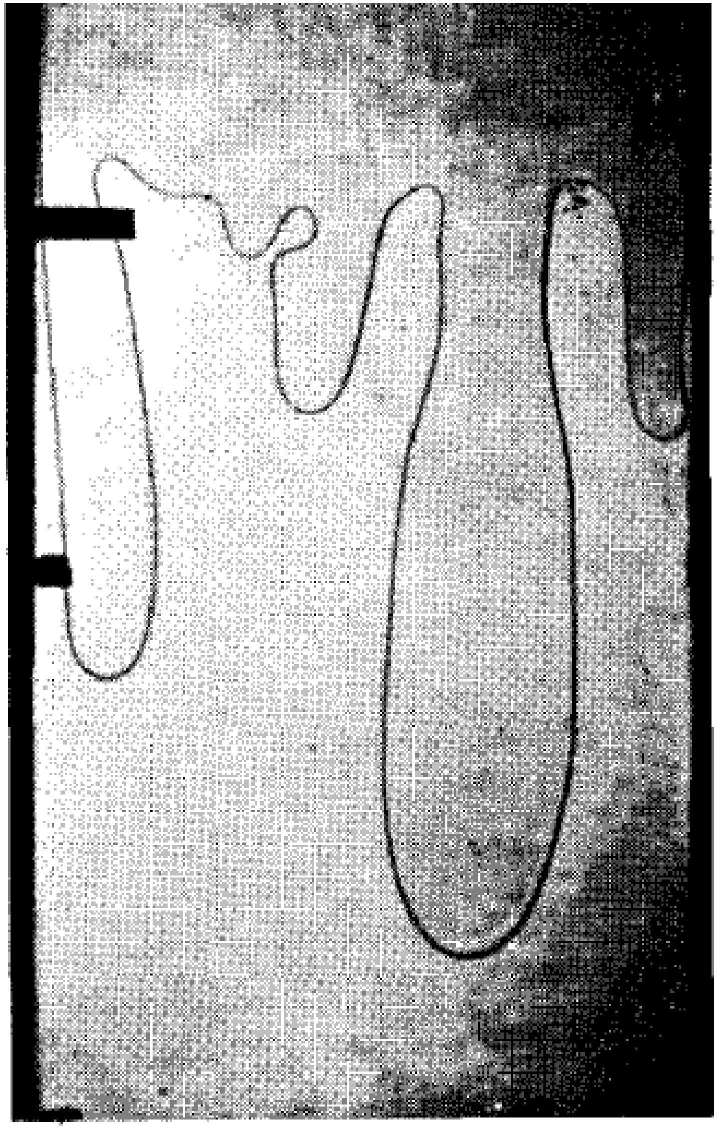
$$k_c^2 = \frac{12U(\eta - \eta')}{\sigma d^2},$$

and it is unstable with respect to perturbations for which $0 < k < k_c$. The perturbations which grow fastest (i.e. for which s_k is a maximum) have $k = k_c/\sqrt{3}$, i.e. a wavelength

$$\lambda = \pi d \sqrt{\frac{\sigma}{U(\eta - \eta')}}. \tag{8.10}$$

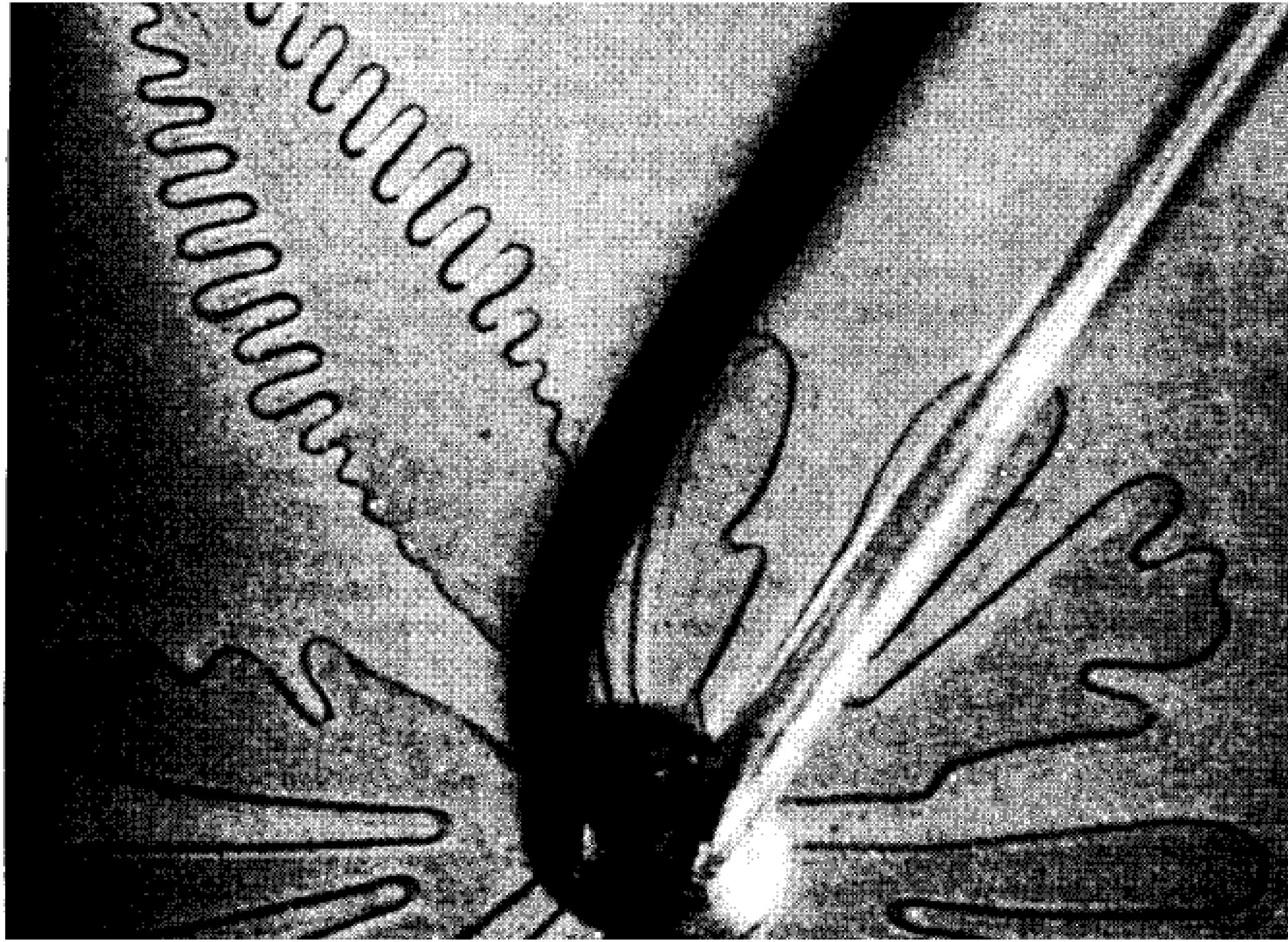
The smallest value of k which is consistent with the boundary conditions at the sides of the cell, where $y = \pm\frac{1}{2}L$, is π/L, and if the cell is so narrow, or if U is so small, that this exceeds k_c then no instabilities can be observed. In the experiments conducted by Saffman and Taylor, however, in which air was used to displace glycerine through a cell whose thickness was about 1 mm, L was 12 cm and the wavelength λ predicted by (8.10) was normally a bit less than 2 cm. Thus they expected to see, when the pressure gradient was first applied, six or seven corrugations develop in the interface over the full width of the cell, and so they did; one of their photographs is reproduced as fig. 8.4(a).



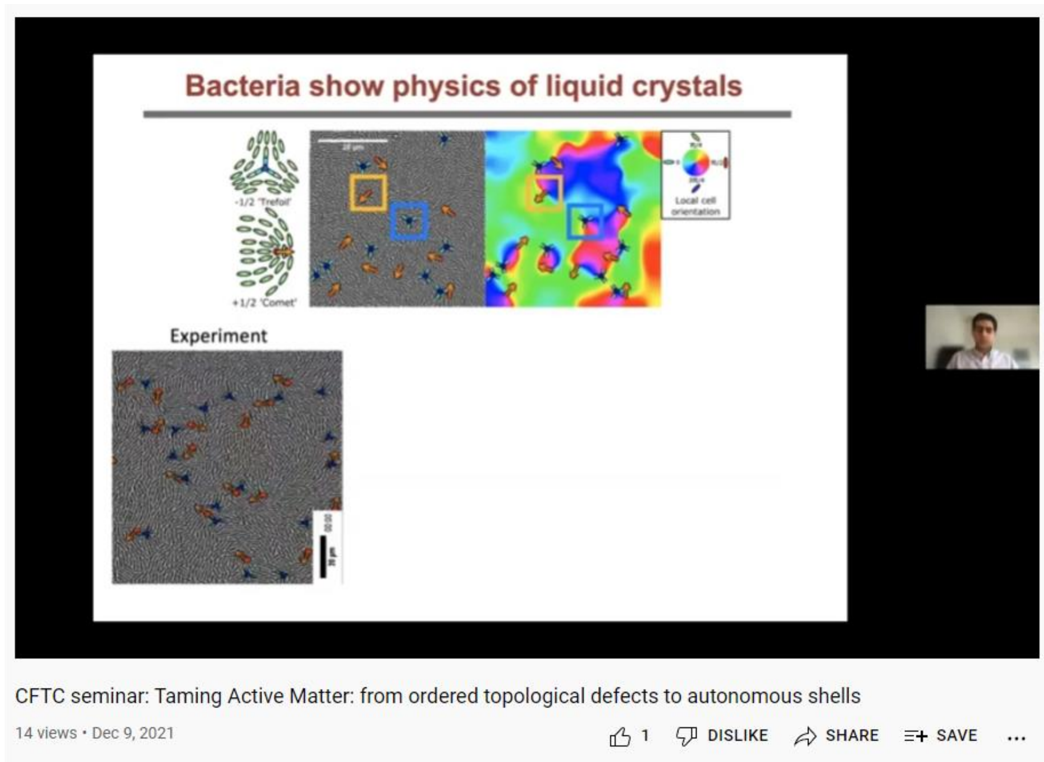


When the corrugations are no longer very small they do not all grow at the same rate, as is shown by fig. 8.4(b). One of the advancing *fingers* of the less viscous fluid tends to get ahead, whereupon it expands sideways and, by doing so, slows down the advance of its competitors. In due course only a single finger survives. It continues to advance at its tip, but it appears to stop expanding sideways when its width reaches half the width of the cell. The tip has a characteristically rounded shape, which Saffman and Taylor were able to explain.

Are the fingers stable and, if not, how do they split up? This question has proved in recent years to be of much greater complexity and interest than Saffman and Taylor could have guessed when their paper on this subject was published in 1958. A partial answer is provided by the two remarkable photographs of fingers spreading radially from a central source which are reproduced in figs. 8.5 and 8.6. The first one shows a number of fingers which are splitting in an irregular and unsurprising way, and one finger which has developed side branches of astonishing regularity; it differs from the others by having a defect at its tip, in the shape of a small gas bubble which has accidentally entered the apparatus and become entrained in the flow. The second photograph shows an even more regular pattern

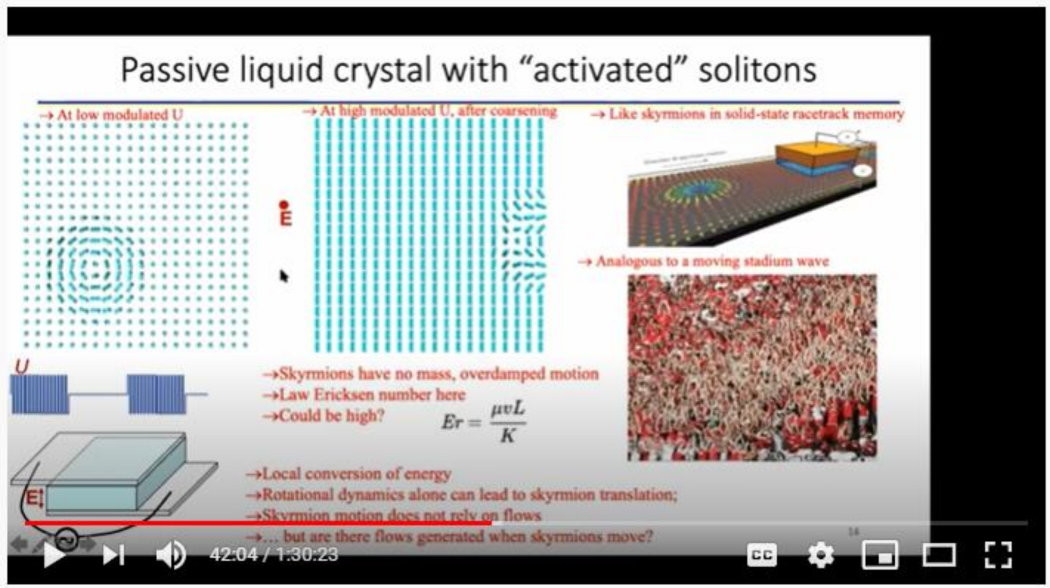


Active nematics



https://www.youtube.com/watch?v=ZpnDwgF3R18

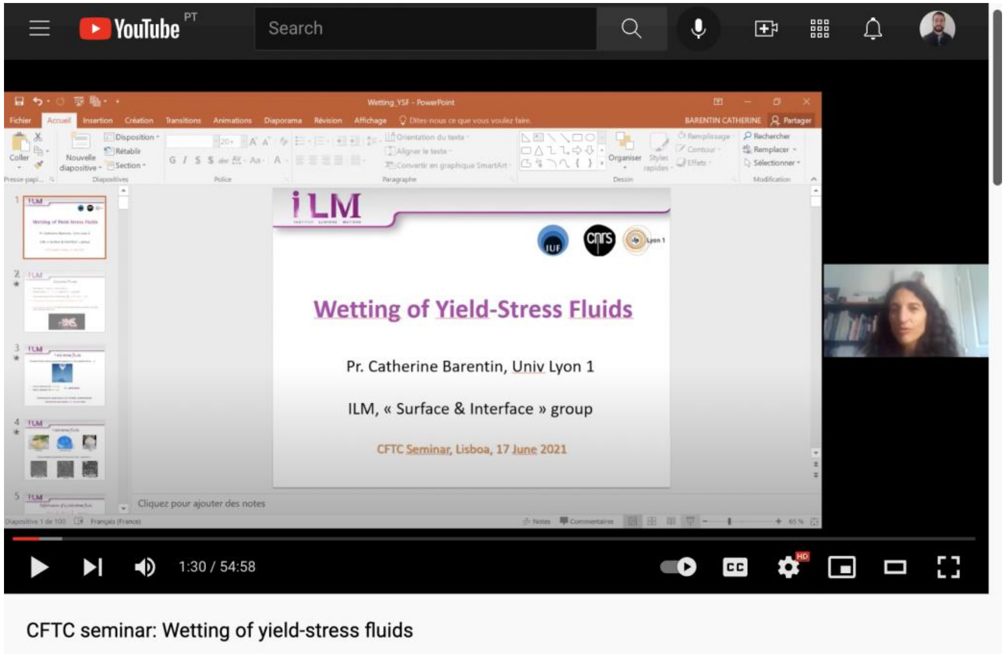
Skyrmions in liquid crystals



CFTC seminar: Topological carnival

https://www.youtube.com/watch?v=rSL7NvFCAR8&t=2084s

Non-newtonian fluids



https://www.youtube.com/watch?v=NkeTh1Vdaew&t=87s

CFTC seminars: https://cftc.ciencias.ulisboa.pt/

Course review

Kinematics

Material derivative

$$\frac{D(\ldots)}{Dt} = \frac{\partial(\ldots)}{\partial t} + \vec{u}\cdot\nabla(\ldots)$$

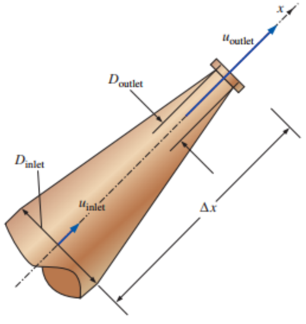
A material derivative is the time derivative of a property following a fluid.

Acceleration

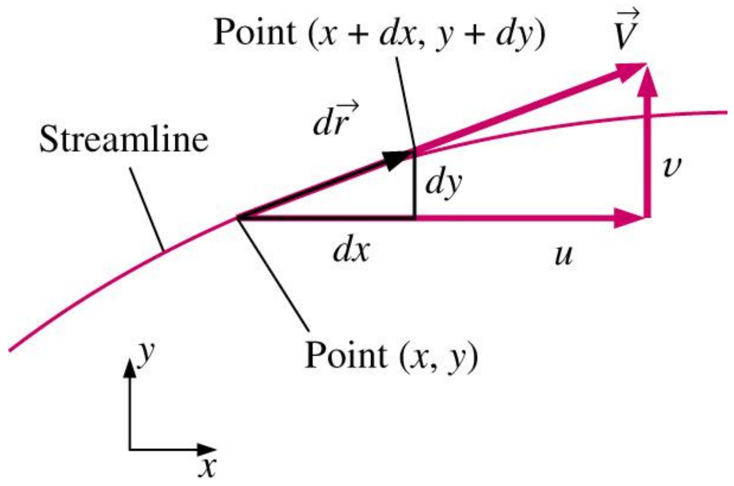
$$\vec{a} = \frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}$$

$\Rightarrow \frac{\partial \vec{u}}{\partial t} = 0$

Steady state does not mean necessarily **a**=0. Ex.:

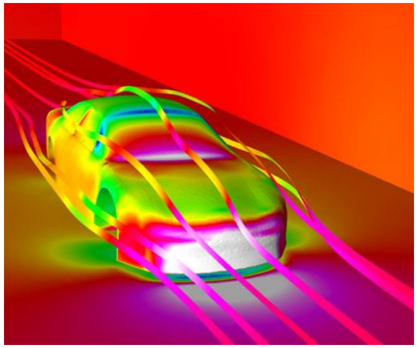


Streamline: is a curve that is everywhere tangent to the *instantaneous* local velocity vector.



$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

NASCAR surface pressure contours and streamlines



Other ways to visualize the flow:

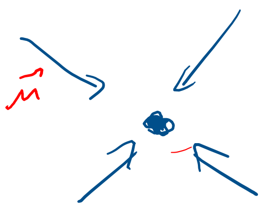
A **Pathline** is the actual path traveled by an individual fluid particle over some time period.

A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.

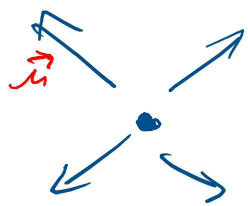
For **steady flow**, streamlines, pathlines, and streaklines are identical.

Continuity equation

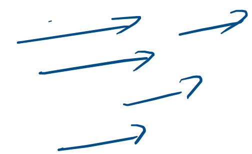
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \,, \quad \text{if } \underbrace{\rho = C_{te}}_{\text{Incompressivel}} \Rightarrow \nabla \cdot \vec{u} = 0$$



$\nabla \cdot \vec{u} < 0$



$\nabla \cdot \vec{u} > 0$

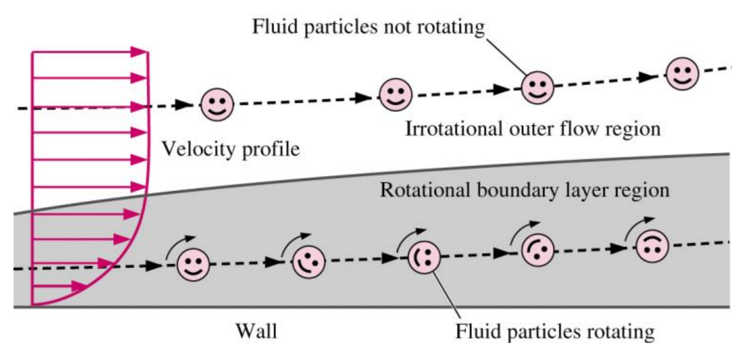


$\nabla \cdot \vec{u} = 0$

Vorticity

$$\vec{w} = \nabla \times \vec{u}$$

Boundary layer:



Euler equation: for incompressible and inviscid fluids.

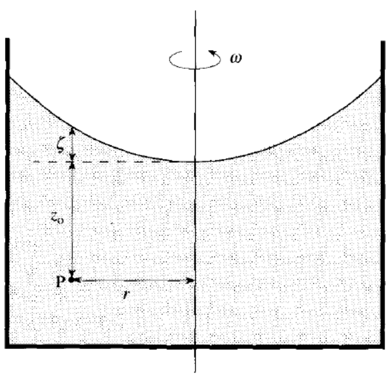
$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \boldsymbol{g},$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right)$$

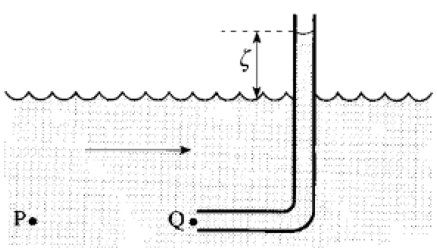
$(\nabla \wedge \boldsymbol{u}) = \vec{\omega}$; $\chi = gz$

$\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right) = c_{te}$ if $\vec{\omega} = 0$; $\rho \frac{\partial \vec{u}}{\partial t} = 0$

$\vec{\omega} \neq 0$ (Euler)



$\vec{\omega} = 0$ (Bernoulli)



Potential flow. For irrotational flows in Euler fluids.

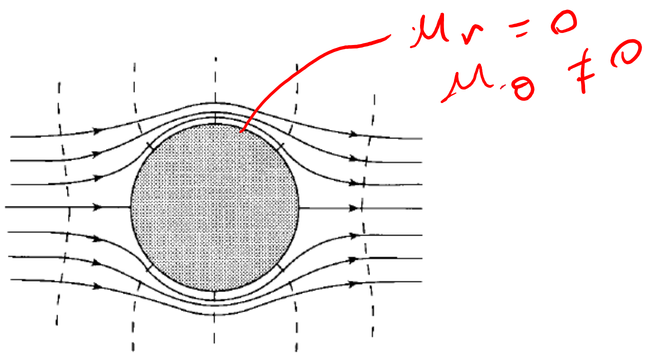
$$\vec{\omega} = \nabla \times \vec{u} = 0 \quad \Rightarrow \quad \vec{u} = \nabla \phi$$

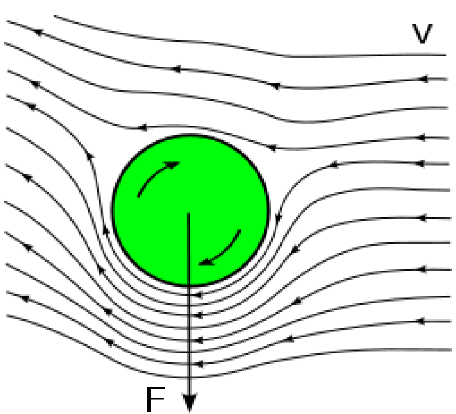
$$\nabla \cdot \vec{u} = 0 \quad \Rightarrow \quad \nabla \cdot \nabla \phi = \nabla^2 \phi = 0$$

In this case, the pressure is given by the Bernoulli equation.

Kelvin circulation theorem: An ideal fluid that is vorticity free at a given instant is vorticity free at all times.

Flow around a sphere: the drag and lift forces are zero for an ideal fluid.





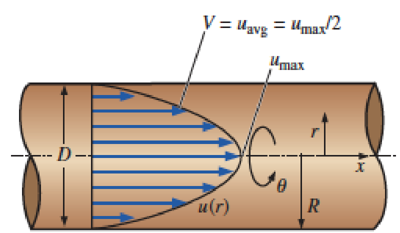
Navier-Stokes: incompressible viscous fluids.

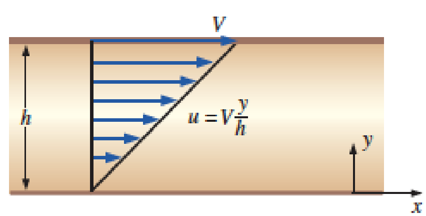
Newtonian fluids, defined as fluids for which the shear stress is linearly proportional to the shear strain rate.

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{\nabla p}{\rho} + \vec{g} + \nu\nabla^2\vec{u}$$

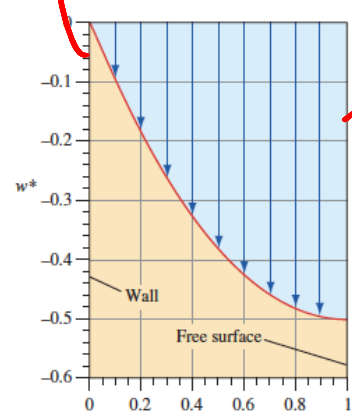
Boundary conditions. 1) no-slip: at the surface, the velocity of the liquid and solid are the same. 2) Interface BC: at the interface, the velocity and the shear-stress of the two fluid are the same. 3) Fre(e) surface BC: at the free surface, the shear stress is zero.

1)





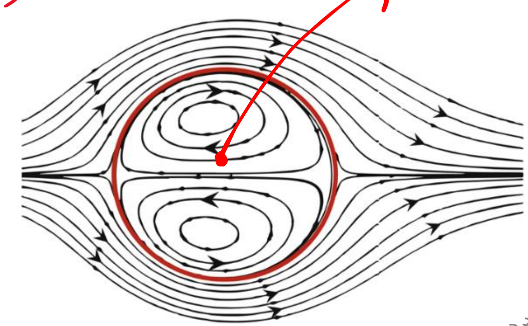
3) $\vec{u} = 0$



$\tau = 0$

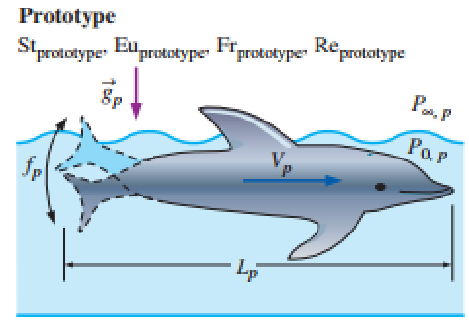
2) $f(r) = \frac{A}{r} + \dots$, $A = 0$

$\vec{u}$: finite

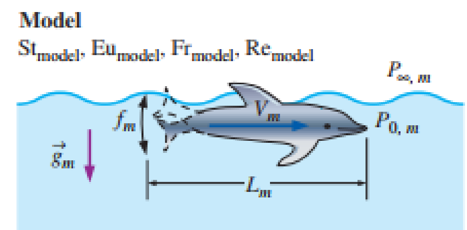


Nondimensionalized Navier–Stokes:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*$$



Since there are four dimensionless parameters, dynamic similarity between a model and a prototype requires all four of these to be the same for the model and the prototype ($\mathrm{St}_{model} = \mathrm{St}_{prototype}$, $\mathrm{Eu}_{model} = \mathrm{Eu}_{prototype}$, $\mathrm{Fr}_{model} = \mathrm{Fr}_{prototype}$, and $\mathrm{Re}_{model} = \mathrm{Re}_{prototype}$).

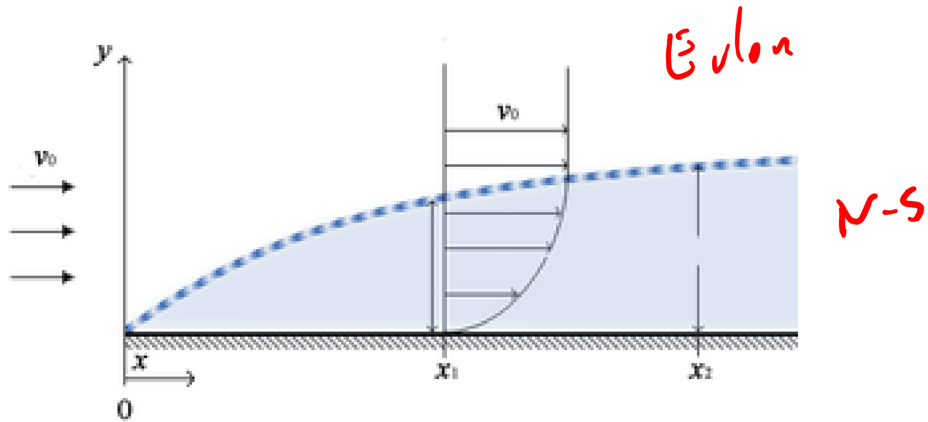


Approximate Navier–Stokes equation for creeping flow: $\vec{\nabla}P \cong \mu\nabla^2\vec{V}$ Roll!

Drag force on a sphere in creeping flow: $F_D = 3\pi\mu VD$

Reversibility of the Stokes equation and the swimming at the microscale.

Boundary layer. Separates viscous and inviscid flows close to a solid surface.



Assumptions to obtain the BL equations

Boundary layer equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = U\dfrac{dU}{dx} + \nu\dfrac{\partial^2 u}{\partial y^2} \end{cases}$$

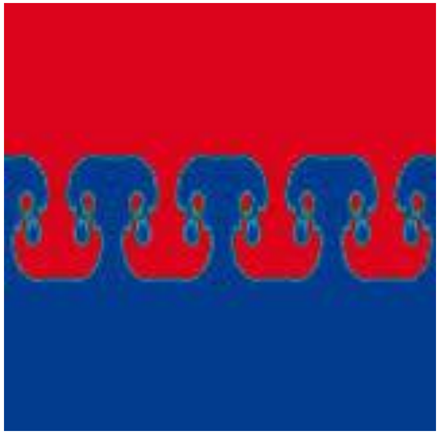
Boundary conditions in the flat plate problem.

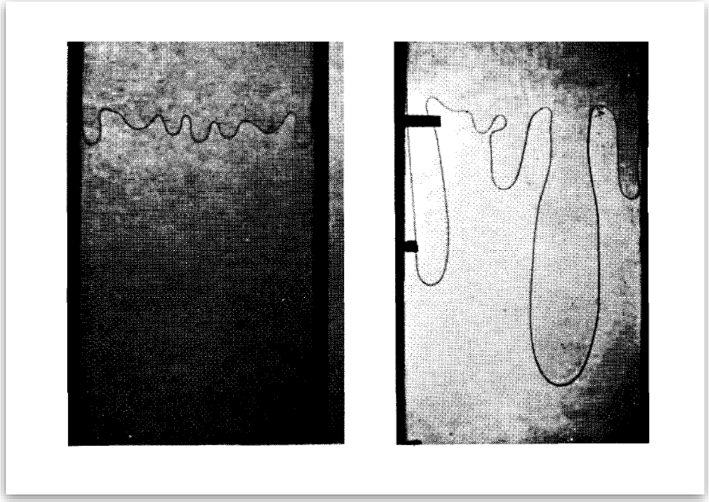
How to calculate the vorticity equation and its interpretation in simple cases

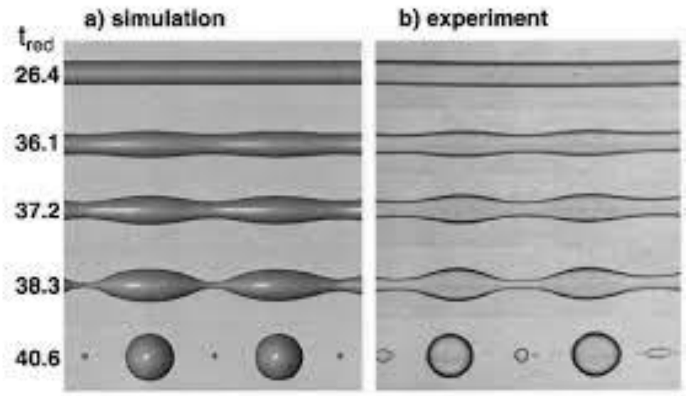
To find what difference viscosity makes, we need to repeat the above analysis using the Navier–Stokes equation as our starting point, rather than the Euler equation. The viscous term on the left-hand side of (6.25) is $-\eta\nabla \wedge \boldsymbol{\Omega}$, and the curl of this, since $\nabla\cdot\boldsymbol{\Omega} = 0$, is $\eta\nabla^2\boldsymbol{\Omega}$. Hence we now have

$$\frac{D\boldsymbol{\Omega}}{Dt} = (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{u} + \frac{\eta}{\rho}\nabla^2\boldsymbol{\Omega}. \quad (7.3)$$

Instabilities







How to find the critical conditions for the instability (marginal instability) and which mode grows faster.

Why does the instabilities happen in each case? Ex.: physical mechanism in the Rayleigh-Taylor instability.

Exame tipo

Questão 1

Euler equation: for incompressible and inviscid fluids.

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \boldsymbol{g},$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right)$$

gz

$(\nabla \wedge \boldsymbol{u}) = \vec{\omega}$

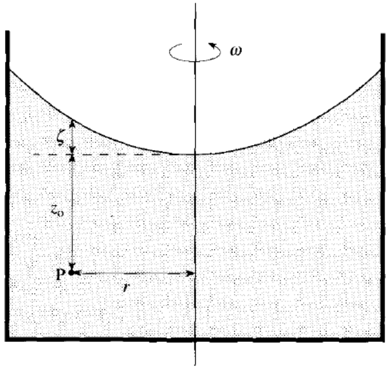
$\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right) = C_{te}$ if $\vec{\omega} = 0$

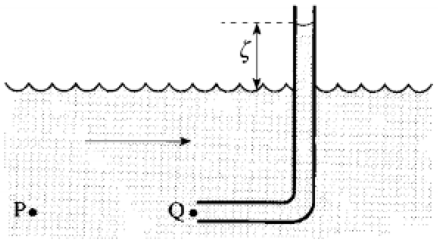
$\rho \frac{\partial \vec{u}}{\partial t} = 0$

$\Rightarrow \vec{u} \cdot \nabla H = 0$

$\vec{\omega} \neq 0$ (Euler)

$\vec{\omega} = 0$ (Bernoulli)





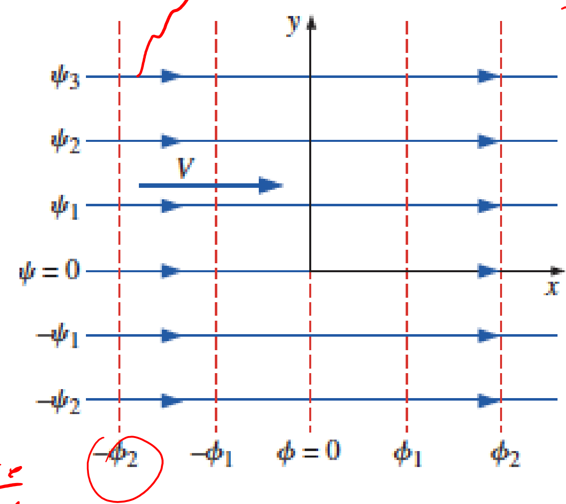
Questão 2

Uniform (free) stream

Pot. de escoamento — *função corrente*

Uniform stream: $u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} = V \qquad v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} = 0$

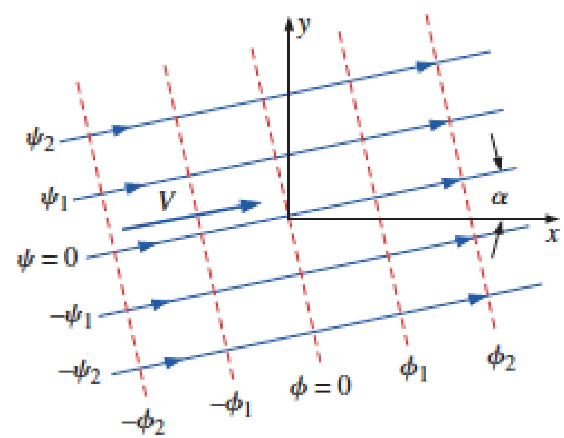
$$\phi = Vx + f(y) \quad \rightarrow \quad v = \frac{\partial \phi}{\partial y} = f'(y) = 0 \quad \rightarrow \quad f(y) = \text{constant}$$



Velocity potential function for a uniform stream: $\phi = Vx$ $= C_{te} \Rightarrow x = \dfrac{C_{te}}{V}$

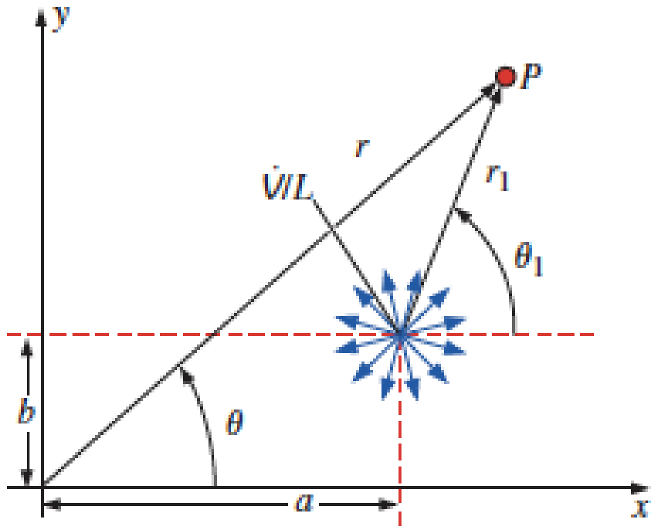
Stream function for a uniform stream: $\psi = Vy$ $= C_2 \Rightarrow y = \dfrac{C_2}{V}$

Uniform stream: $\phi = Vr\cos\theta \qquad \psi = Vr\sin\theta$



Uniform stream inclined at angle α: $\begin{cases} \phi = V(x\cos\alpha + y\sin\alpha) \\ \psi = V(y\cos\alpha - x\sin\alpha) \end{cases}$

Line source or sink at an arbitrary point



Line source at point (a, b):

$$\phi = \frac{\dot{V}/L}{2\pi} \ln r_1 = \frac{\dot{V}/L}{2\pi} \ln \sqrt{(x-a)^2 + (y-b)^2}$$

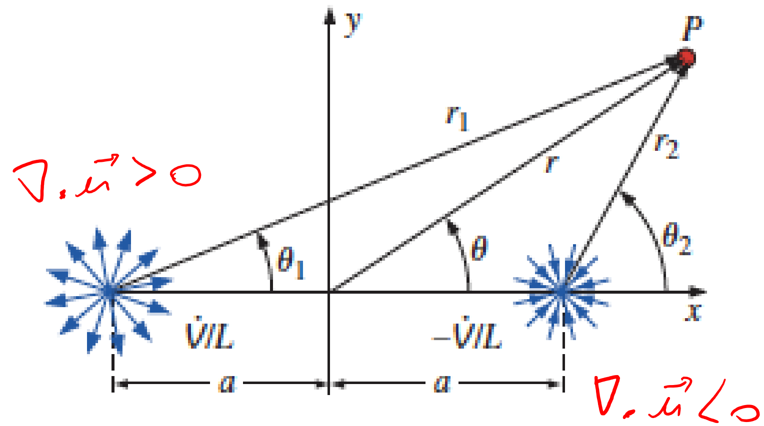
$$\psi = \frac{\dot{V}/L}{2\pi} \theta_1 = \frac{\dot{V}/L}{2\pi} \arctan \frac{y-b}{x-a}$$

Superposition of a source and sink of equal strength

Line source at $(-a, 0)$: $\psi_1 = \frac{\dot{V}/L}{2\pi}\theta_1$ where $\theta_1 = \arctan\frac{y}{x+a}$

Similarly for the sink,

Line sink at $(a, 0)$: $\psi_2 = \frac{-\dot{V}/L}{2\pi}\theta_2$ where $\theta_2 = \arctan\frac{y}{x-a}$



Composite stream function: $\psi = \psi_1 + \psi_2 = \frac{\dot{V}/L}{2\pi}(\theta_1 - \theta_2)$

Final result, Cartesian coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ay}{x^2 + y^2 - a^2}$

Final result, cylindrical coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ar\sin\theta}{r^2 - a^2}$

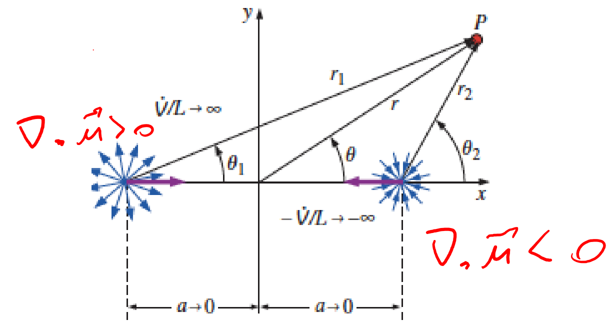
Using

$$\arctan(u) \pm \arctan(v) = \arctan\left(\frac{u \pm v}{1 \mp uv}\right) \pmod{\pi}, \quad uv \neq 1 .$$

Doublet: line source and sink close to origin

We have seen before that

Composite stream function: $\psi = \dfrac{-\dot{V}/L}{2\pi} \arctan \left[\dfrac{2ar\sin\theta}{r^2 - a^2} \right]$



By Taylor expanding the arctan around zero:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$Tg^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

$a \ll r$

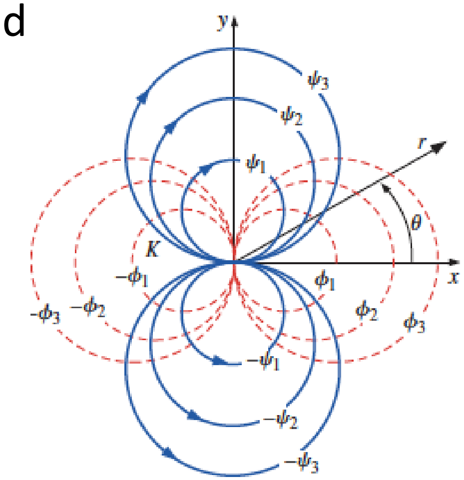
Stream function as $a \to 0$: $\psi \to \dfrac{-a(\dot{V}/L) r \sin\theta}{\pi(r^2 - a^2)}$

Doublet: line source and sink close to origin

Let a tend to zero at constant doublet strength K, to find

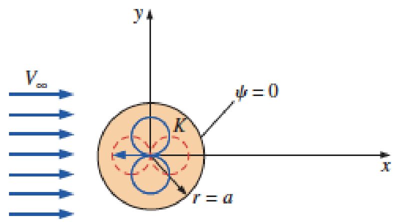
Doublet along the x-axis: $$\psi = \frac{-a(\dot{V}/L)}{\pi}\frac{\sin\theta}{r} = -K\frac{\sin\theta}{r}$$

Doublet along the x-axis: $$\phi = K\frac{\cos\theta}{r}$$



Streamlines (solid) and equipotential lines (dashed) for a doublet of strength K located at the origin in the xy-plane and aligned with the x-axis.

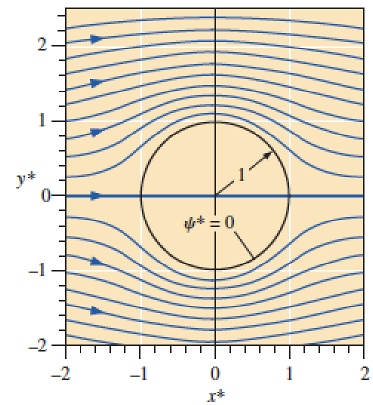
Superposition of a uniform stream and a doublet: Flow over a circular cylinder



Superposition: $$\psi = V_\infty r \sin\theta - K\frac{\sin\theta}{r}$$

For convenience we set $\psi = 0$ when $r = a$

Doublet strength: $$K = V_\infty a^2$$

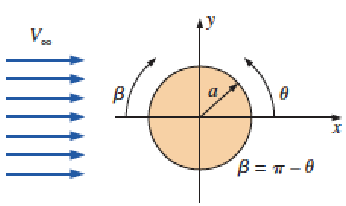


Alternate form of stream function: $$\psi = V_\infty \sin\theta\left(r - \frac{a^2}{r}\right)$$

$$\psi^* = \sin\theta\left(r^* - \frac{1}{r^*}\right)$$

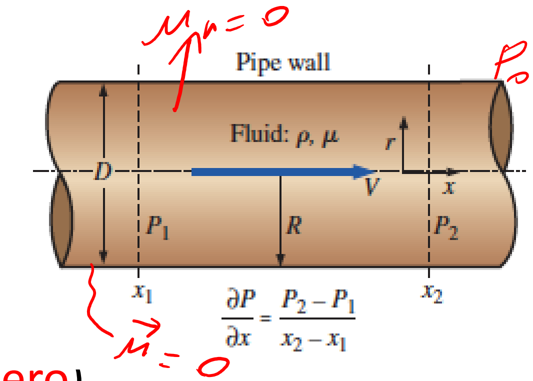
Nondimensional streamlines: $$r^* = \frac{\psi^* \pm \sqrt{(\psi^*)^2 + 4\sin^2\theta}}{2\sin\theta}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta\left(1 - \frac{a^2}{r^2}\right) \quad u_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta\left(1 + \frac{a^2}{r^2}\right)$$



Questão 3

Flow in a round pipe: Poiseuille



1 The pipe is infinitely long in the x-direction.

2 The flow is steady (all partial time derivatives are zero).

3 This is a parallel flow (the r-component of velocity, u_r, is zero).

4 The fluid is incompressible and Newtonian with constant properties, and the flow is laminar.

5 A constant pressure gradient is applied in the x-direction such that pressure changes linearly with respect to x.

6 The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all partial derivatives with respect to θ are zero.

7 We ignore the effects of gravity.

8 The first boundary condition comes from imposing the no slip condition at the pipe wall: (1) at $r = R, \vec{V} = 0$.

9 The second boundary condition comes from the fact that the centerline of the pipe is an axis of symmetry: (2) at $r = 0$, $\frac{\partial u}{\partial x} = 0$. Alternatively: the velocity is finite at the center.

Continuity: $\nabla \cdot \vec{u} = 0$

$u_r = 0$, $u_\theta = 0$

$$\frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial u}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Result of continuity: $u = u(r)$ only

NS u:

$$\rho\left(\frac{\partial u}{\partial t} + u_r\frac{\partial u}{\partial r} + \frac{u_\theta}{r}\frac{\partial u}{\partial \theta} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2}\right)$$

$\vec{g} = 0$, $\frac{\partial}{\partial \theta}(\dots) = 0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}$$

NS p:

r-momentum: $\frac{\partial P}{\partial r} = 0$

Result of r-momentum: $P = P(x)$ only $\Rightarrow P = \frac{\partial P}{\partial x} \cdot x + P_0$

Integration of NS for u:

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1$$

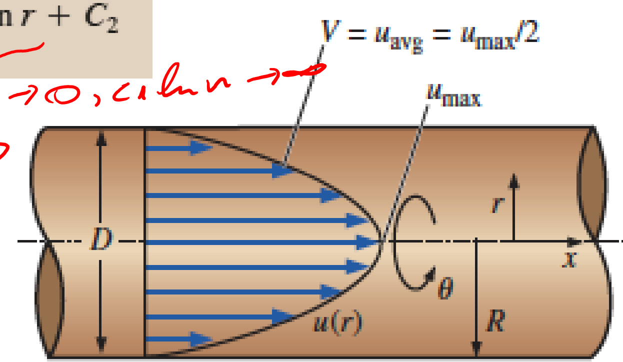
$$u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2$$

$u(r = R) = 0$

$r \to 0$, $C_1 \ln r \to \infty$ $\Rightarrow C_1 = 0$

Axial velocity:

$$u = \frac{1}{4\mu}\frac{dP}{dx}(r^2 - R^2)$$



Poiseuille's law for the flow rate

Maximum axial velocity: $$u_{\max} = -\frac{R^2}{4\mu}\frac{dP}{dx}$$

$$\dot{V} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} ur\,dr\,d\theta = \frac{2\pi}{4\mu}\frac{dP}{dx}\int_{r=0}^{R}(r^2 - R^2)r\,dr = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}$$

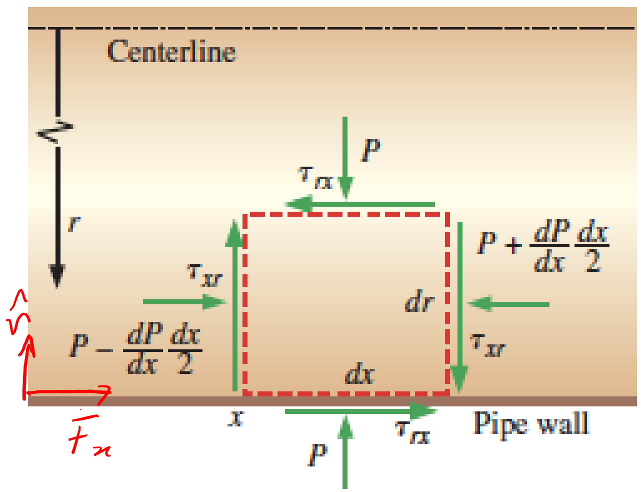
Average axial velocity: $$V = \frac{\dot{V}}{A} = \frac{(-\pi R^4/8\mu)\,(dP/dx)}{\pi R^2} = -\frac{R^2}{8\mu}\frac{dP}{dx}$$

Viscous shear force

The stress tensor is

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rx} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta x} \\ \tau_{xr} & \tau_{x\theta} & \tau_{xx} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mu \dfrac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \mu \dfrac{\partial u}{\partial r} & 0 & 0 \end{pmatrix}$$

Viscous shear stress at the pipe wall: $\tau_{rx} = \mu \dfrac{du}{dr} = \dfrac{R}{2}\dfrac{dP}{dx}$

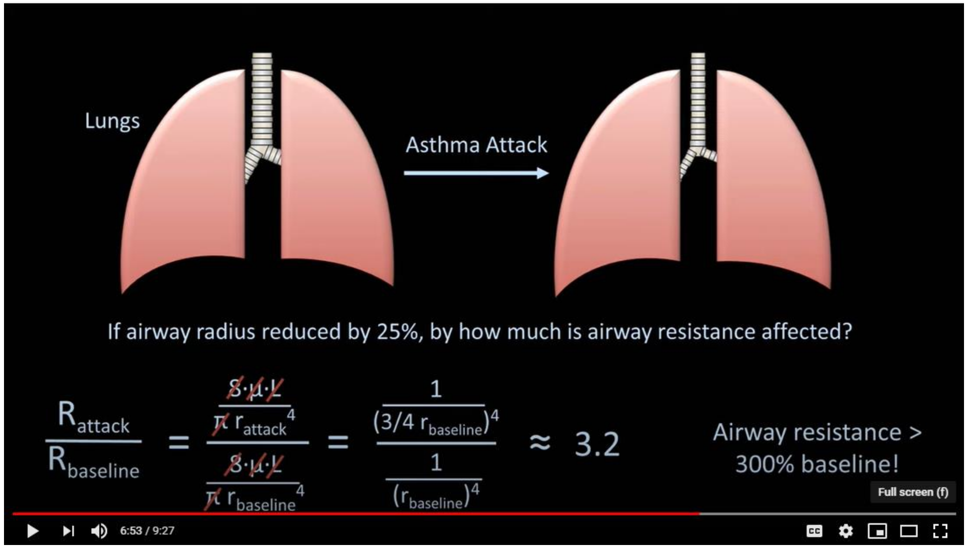


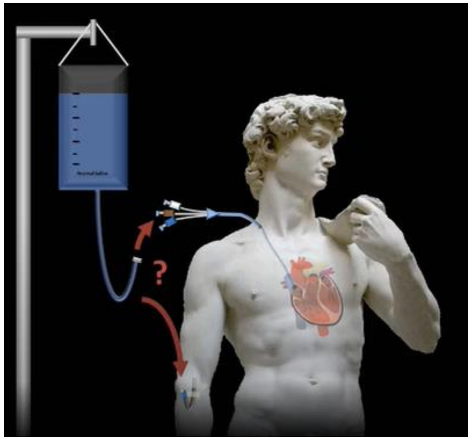
For flow from left to right, dP/dx is negative, so the viscous shear stress on the bottom of the fluid element at the wall is in the direction opposite to that indicated in the figure. (This agrees with our intuition since the pipe wall exerts a retarding force on the fluid.) The shear force per unit area on the wall is equal and opposite to this; hence,

Viscous shear force per unit area acting on the wall: $\dfrac{\vec{F}}{A} = -\dfrac{R}{2}\dfrac{dP}{dx}\vec{i}$

Viscosity and Poiseuille's Law:

https://www.youtube.com/watch?v=wTnI_kfPBhQ





Force balance

Navier-Stokes equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}}_{=0} = -\frac{\nabla p}{\rho} + \vec{g} + \nu \nabla^2 \vec{u}$$

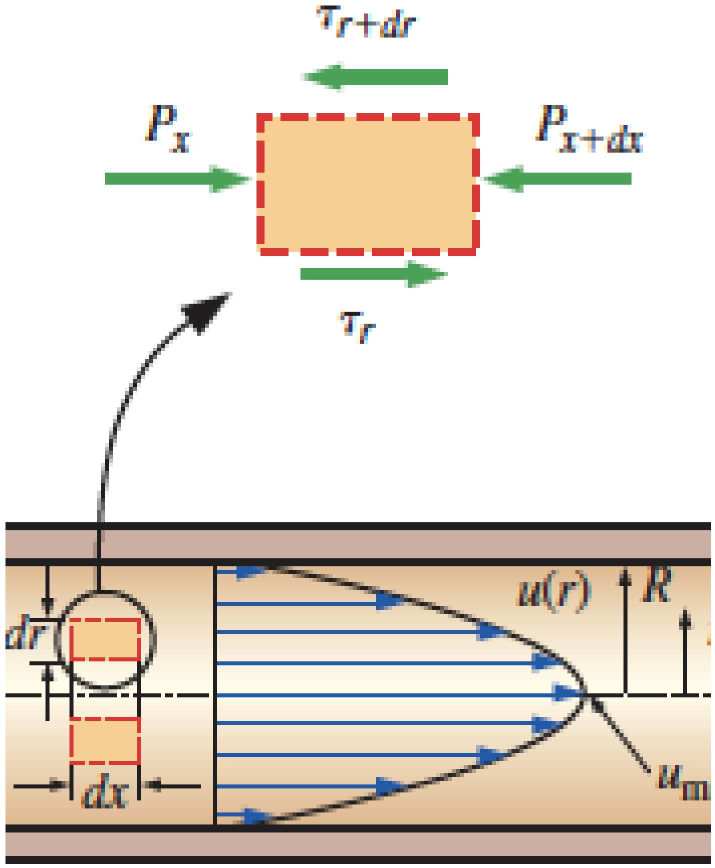
In most of the previous examples, the acceleration of the fluid elements is zero. It means that the viscous force balance the external force (e.g., gravity) or pressure gradients in such a way that the sum of forces acting on a fluid element is zero.

Sec. 8.4, Çengel

Alternative derivation for flow in a circular pipe

Obtain the momentum equation by applying a momentum balance to a differential volume element, and we obtain the velocity profile by solving it.

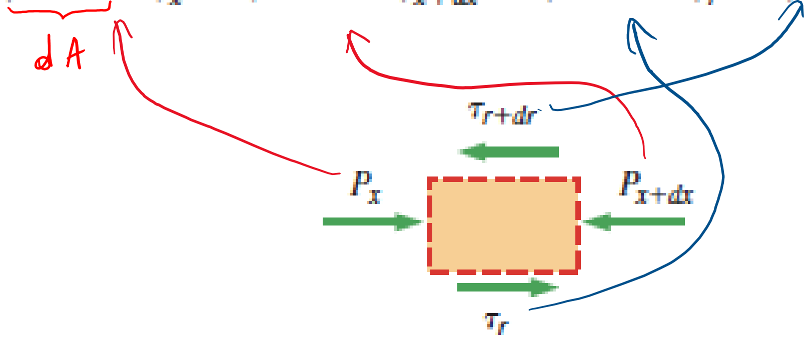
Free-body diagram of a ring-shaped differential fluid element of radius r, thickness dr, and length dx oriented coaxially with a horizontal pipe in fully developed laminar flow.

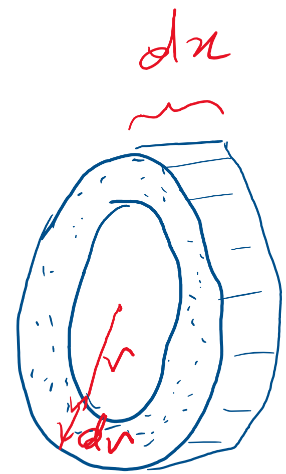


In fully developed laminar flow the axial velocity is, u= u(r). There is no motion in the radial direction. There is no acceleration (check: calculate the acceleration and verify that it is zero).

- Consider a ring-shaped differential volume element of radius r, thickness dr, and length dx oriented coaxially with the pipe.
- The volume element involves only pressure and viscous effects and thus the pressure and shear forces must balance each other. The pressure force acting on a submerged plane surface is the product of the pressure at the centroid of the surface and the surface area. A force balance on the volume element in the flow direction (x) gives

$$(2\pi r\, dr\, P)_x - (2\pi r\, dr\, P)_{x+dx} + (2\pi r\, dx\, \tau)_r - (2\pi r\, dx\, \tau)_{r+dr} = 0$$





Force balance implies

$$(2\pi r\, dr\, P)_x - (2\pi r\, dr\, P)_{x+dx} + (2\pi r\, dx\, \tau)_r - (2\pi r\, dx\, \tau)_{r+dr} = 0 \quad \div (2\pi\, dx\, dr)$$

$$r\,\frac{\overbrace{P_{x+dx} - P_x}^{dP}}{dx} + \frac{\overbrace{(r\tau)_{r+dr} - (r\tau)_r}^{d(r\tau)}}{dr} = 0$$

$$r\,\frac{dP}{dx} + \frac{d(r\tau)}{dr} = 0$$

and substituting the stress (component rz): $\tau_{rz} = -\mu(du/dr)$ we find

$$\boxed{\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx}}$$

Same equation obtained with NS: $\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}$

Recall

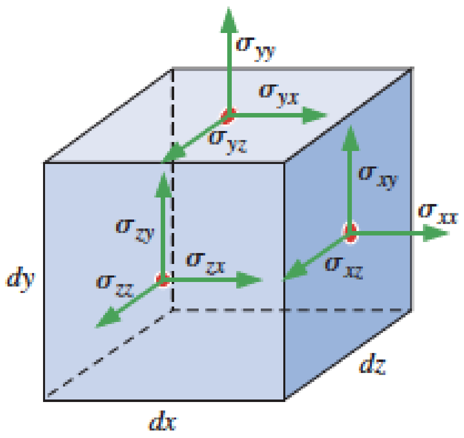
Deviatoric stress tensor

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

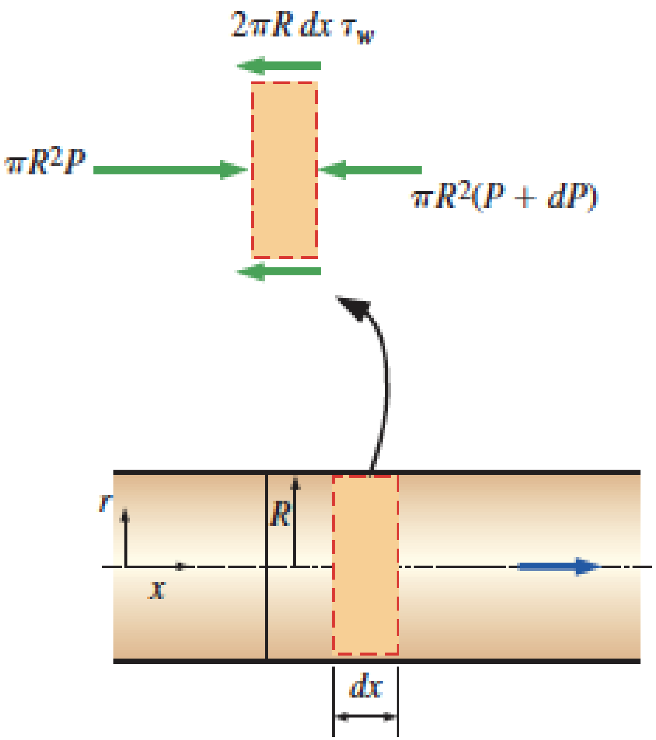
$$= \begin{pmatrix} 2\mu \frac{\partial u_r}{\partial r} & \mu\left[r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] & \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ \mu\left[r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] & 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) & 2\mu \frac{\partial u_z}{\partial z} \end{pmatrix}$$

Stress tensor

$$\sigma_{ij} = -P\,\delta_{ij} + \tau_{ij}$$



Different fluid element (r from 0 to R)



Force balance:

$$\pi R^2 P - \pi R^2(P + dP) - 2\pi R\,dx\,\tau_w = 0$$

Simplifying:

$$\frac{dP}{dx} = -\frac{2\tau_w}{R}$$

Separation of variables implies that the pressure gradient is constant $\frac{dP}{dx} = -\frac{2\tau_w}{R}$

The velocity profile is obtained by integration and use of the boundary conditions:

$$u(r) = \frac{r^2}{4\mu}\left(\frac{dP}{dx}\right) + C_1 \ln r + C_2$$

(...) u(r = R) = 0

= 0 (u(r = 0) e' finito)

$$u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)$$

The average velocity is

$$V_{avg} = \frac{2}{R^2}\int_0^R u(r) r\, dr = \frac{-2}{R^2}\int_0^R \frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right) r\, dr = -\frac{R^2}{8\mu}\left(\frac{dP}{dx}\right)$$

In terms of which the profile becomes

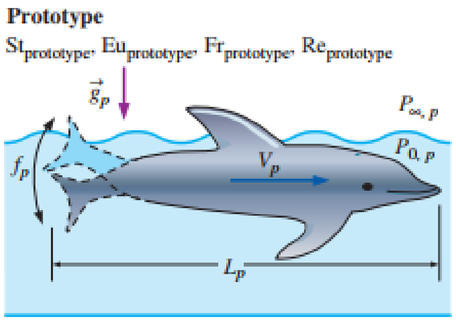
$$u(r) = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)$$

r = 0 => u(r) = u_MAX = 2 V_avg

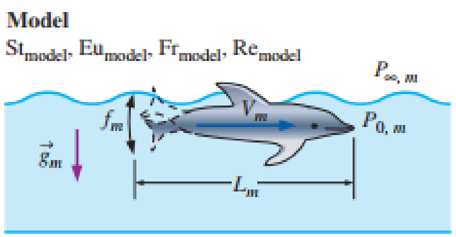
Questão 4

Nondimensionalized Navier–Stokes:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*$$



Since there are four dimensionless parameters, dynamic similarity between a model and a prototype requires all four of these to be the same for the model and the prototype ($\mathrm{St}_{model} = \mathrm{St}_{prototype}$, $\mathrm{Eu}_{model} = \mathrm{Eu}_{prototype}$, $\mathrm{Fr}_{model} = \mathrm{Fr}_{prototype}$, and $\mathrm{Re}_{model} = \mathrm{Re}_{prototype}$).



Approximate Navier–Stokes equation for creeping flow: $\vec{\nabla}P \cong \mu\nabla^2\vec{V}$

Drag force on a sphere in creeping flow: $F_D = 3\pi\mu VD$