Cosmologia Física

Homework 5

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Exercise 1: Dark matter non-linear perturbations

1.1) Consider the halo mass function

$$\frac{dn}{dM} = -\left(\frac{2}{\pi}\right)^{1/2} \frac{\Omega_m \rho_c}{M} \frac{\delta_c}{\sigma_M^2} \frac{d\sigma_M}{dM} \exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right).$$

The halo mass function is valid for a wide range of spherical halos of sizes R and corresponding masses $M = (4/3)\pi R^3 \bar{\rho}$. The correlation function of those halos is measured to be $\xi \sim r^{-(3+n)}$, where n is the power spectrum index (remember exercise 2 of Homework 3). The clustering amplitude σ_R (also called σ_M) of a scale of size R and mass M has the same dependence in R as the correlation function, i.e.,

$$\sigma_R^2 = \sigma_0^2 \left(\frac{R}{R_0}\right)^{-(3+n)} = \sigma_8^2 \left(\frac{R}{R_8}\right)^{-(3+n)} ,$$

where R_0 or R_8 are just examples of reference scales (any scale can be used for reference).

a) What is the dimension (i.e. the units) of the mass function?

b) Show that the amplitude in function of mass is

$$\sigma_M = \sigma_0 \left(\frac{M}{M_0}\right)^{-(3+n)/6} .$$

c) Show that the mass function given above can be written as

$$\frac{dn}{dM} = \frac{1}{\sqrt{\pi}} \Omega_m \rho_c \frac{3+n}{3} \frac{1}{M^2} \left(\frac{M}{M_0}\right)^{\frac{3+n}{6}} \exp\left[-\left(\frac{M}{M_0}\right)^{\frac{3+n}{3}}\right] .$$

d) As you realized in your derivation, the result in c) is only valid for one particular reference scale M_0. What is the name and physical meaning of that scale? Compute its numerical value in terms of the reference mass M_8, using the standard values for the linear critical density for virialization ($\delta_c = 1.68$), for the clustering amplitude ($\sigma_8 = 0.9$) and power spectrum index ($n = -1.3$).

e) Compute the total mass density in the Universe (in the assumption that all mass is contained in halos).
Hint: Use the mass function derived in c).

Exercise 2: Gravitational lensing

2.1) The light from a source galaxy passes through an over-dense region on the line-of-sight that acts as a lens. The lens creates a distorted image of the source, such that each point $\vec{\beta}$ of the source appear at a position $\vec{\theta}$ in the image. The linear mapping from source to image is given by the amplification matrix (also called distortion),

$$A_{ij}(\theta) = \frac{\partial \beta_i}{\partial \theta_j} = \left(\delta_{ij} - \frac{\partial \alpha_i}{\partial \theta_j} \right) = \begin{bmatrix} 1 - k - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - k + \gamma_1 \end{bmatrix}.$$

a) Compute the solution $\vec{\theta} = f(\vec{\beta})$ for the case where only k is nonzero ($\gamma_1 = \gamma_2 = 0$) and is constant across the image. (Note: set any integration constant to zero).

b) From the result of a) what is the shape of the image of a circular source?

c) Consider a galaxy at $z = 0.8$ in the concordance Universe ($\mathrm{h} = 0.7, \Omega_\mathrm{m} = 0.3, \Omega_\Lambda = 0.7$). The line-of-sight contains a lens with $k = 0.3$ and no shear. Assume the observed galaxy has a radius of 1 arcmin. What is the physical size of the galaxy? (Note: use the on-line cosmology calculator if needed).

d) Consider now a circular source of radius β_0 and a lens on the line-of-sight with only γ_1 ($k = \gamma_2 = 0$). Compute the solution $\vec{\theta} = f(\vec{\beta})$ for this case, and show that the image of the circular source is an ellipse. Write the expressions for the major and minor axis of that ellipse as function of γ_1 and β_0.

2.2) The ellipticity of a two-dimensional shape (e_1, e_2) is defined from the second-order moments Q_{ij} of the shape,

$$e = \left(\frac{Q_{11} - Q_{22}}{Q_{11} + Q_{22}}, \frac{2Q_{12}}{Q_{11} + Q_{22}} \right).$$

The moments of the lensed image form a tensor and are transformed from the moments of the source with the standard matrix transformation,

$$Q^s = A(\theta)\, Q\, A^T(\theta),$$

where A is the distortion matrix.

a) Compute the transformation of the ellipticity, obtaining an expression relating the ellipticity of the image (e) to the ellipticity of the source (e^s), for the case of very small distortions (i.e. assume that $k << 1$, $\gamma << 1$, $(1 - k) \approx 1$, $\gamma k \approx 0$, etc).

Exercise 3: Supernova surveys

The distance modulus, $\mu = 5\log_{10} D_L + 25$, depends on the cosmological parameters through the luminosity distance D_L. In the following, consider that D_L is given by the approximation:

$$D_L = \frac{c}{H_0}\left[z + \frac{z^2}{2}\left(1 + \Omega_\Lambda - \frac{\Omega_\mathrm{m}}{2}\right)\right].$$

3.1)

a) Is the observable μ more sensitive to Ω_m or to Ω_Λ?

b) μ depends on a combination of the two parameters of the model. This implies that a measurement of μ correlates the parameters. The estimated parameter values become correlated or anti-correlated?

c) In a flat model (and considering only a fluid with two components) the two parameters are not independent: are they correlated or anti-correlated?

d) From the results of b) and c) would you say that assuming a flat model in the parameter estimation procedure is a strong prior that has a large impact on the results or a weak prior with negligible impact?

3.2) Consider a very small supernova survey that detected 3 supernovae and measured their distance modulus with uncorrelated errors. The data vector obtained is: distance modulus $\mu = 42.10 \pm 0.25$ for the object SN 2002dc at z=0.475; $\mu = 43.14 \pm 0.21$ for SN 2003bd at z=0.67; and $\mu = 44.25 \pm 0.14$ for SN 2001hb at z=1.0. (These values are taken from the SNLS survey).

Consider also the Fisher matrix for the distance modulus, which is given by

$$F_{ij} = \sum_z \left[\left(\frac{\partial \mu_z}{\partial p_i}\right)_\mathrm{fid} \frac{1}{\sigma_z^2}\left(\frac{\partial \mu_z}{\partial p_j}\right)_\mathrm{fid}\right]$$

a) Compute the Fisher matrix of this survey, assuming $\Omega_\mathrm{m} = 0.3$ and $\Omega_\Lambda = 0.7$ as the fiducial model. (Note that we do not need to give a fiducial value to the Hubble constant, because the Fisher matrix will be independent of H_0.)

b) Notice that the Fisher matrix just obtained is singular (which does not happen when using the full integral expression for D_L). What is the feature in the $D_L(z)$ approximate formula that is responsable for this behaviour?

c) Being singular, the associated contour ellipse stretches to infinity, and we cannot get finite constraints on the individual parameters. This means that μ only has information about one effective parameter: the combination $\Omega = \Omega_\Lambda - \Omega_\mathrm{m}/2$. Use the Fisher matrix method in just one dimension for the effective parameter Ω, to find the constraint on that parameter. Write the result in the form: $\Omega = \Omega_\mathrm{fid} \pm \sigma_\Omega$.

3.3) Consider again the supernova survey of three supernovae presented above, but now including the stretch bias in the estimator. This implies that the μ estimator needs to be corrected with an additive parameter, i.e.,

$$\mu = 5\log_{10} D_L + 25 + \alpha\,(s-1),$$

where, for simplification, the 3 SNe all have the same stretch factor s, and so the additive term is a constant bias parameter, $b = \alpha\,(s-1)$.

The bias also correlates the measurements of the 3 supernovae, implying that the errors are now given by a non-diagonal covariance matrix:

$$C = \begin{matrix} (0.25)^2 & 0.007 & 0.003 \\ 0.007 & (0.21)^2 & 0.005 \\ 0.003 & 0.005 & (0.14)^2 \end{matrix}$$

a) The impact of the bias is larger at low or at high redshift?

b) Compute the Fisher matrix, for the biased SNe survey on the (Ω, b) parameter space, where $\Omega = \Omega_\Lambda - \Omega_\mathrm{m}/2$, using again the second-order approximation formula for D_L.
Hint: Remember the Fisher matrix is now given by its full form to account for the cross-correlations:

$$F_{ij} = \left(\frac{\partial \mu_z}{\partial p_i}\right)^T_{\mathrm{fid}} C^{-1}_{zz'} \left(\frac{\partial \mu_{z'}}{\partial p_j}\right)_{\mathrm{fid}}$$

c) Compute the marginalized uncertainty of Ω, i.e. marginalize the Fisher matrix over the bias parameter b.

d) Compare this result with the constraint obtained in the unbiased case. Did the uncertainty on the cosmological parameter Ω increase or decrease by correcting the bias? Is this the expected behaviour? Why?

Exercise 4: Dependence on Cosmological Parameters

4.1) Consider the matter power spectrum and the CMB power spectrum.

a) What is the physical meaning of the main peak of the matter power spectrum?

b) What is the physical meaning of the first peak of the CMB angular power spectrum?

4.2) Say if the following sentences are true or false, justifying your answer (no calculations are needed).

With respect to the concordance model:

a) increasing the value of Ω_m, the main peak of the matter power spectrum moves to the right.

b) including an early dark energy model that increases the speed of sound in the baryon-photon plasma, the first peak of the CMB moves to the right.

c) increasing the value of the Hubble constant, there is an effect on the amplitude of the first peak of the CMB.

d) increasing the value of the Hubble constant, the first peak of the CMB moves to the left.

e) decreasing the value of Ω_b, the amplitudes of the odd peaks of the CMB increase with respect to the amplitudes of the even peaks.

f) increasing the value of Ω_m, the amplitude of the first peak of the CMB increases due to the early ISW effect.

g) increasing the value of n_s, the amplitude of the first peak of the CMB increases.

h) increasing the value of n_s, the amplitude of the fifth peak of the CMB increases