EXERCÍCIOS – FOLHA 1

1.1. Determine bases para $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, $\mathcal{R}(\mathbf{A}^{\mathsf{T}})$ e $\mathcal{N}(\mathbf{A}^{\mathsf{T}})$ onde:

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix} \in \mathbb{R}^{3\times 4}$.

(b) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{bmatrix} \in \mathbb{R}^{3\times 5}$.

1.2. Sejam $\mathbf{A} \in \Bbbk^{m\times n}$ e $\mathbf{b} \in \Bbbk^{m\times 1}$. Prove que $\mathbf{b} \in \mathcal{R}(\mathbf{A})$ se e só se $r\big([\mathbf{A}\ \mathbf{b}]\big) = r(\mathbf{A})$.

1.3. Verifique se $\begin{bmatrix} -2 \\ -5 \\ -6 \\ -7 \\ -7 \end{bmatrix} \in \mathcal{R}(\mathbf{A})$ onde $\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 4 \end{bmatrix} \in \mathbb{R}^{5\times 5}$.

1.4. Seja $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \in \Bbbk^{n\times n}$ uma matriz quadrada tal que $\mathcal{N}(\mathbf{A}_1) = \mathcal{R}(\mathbf{A}_2^{\mathsf{T}})$. Prove que $\mathbf{A}$ é invertível.

1.5. Prove que, para quaisquer $\mathbf{A} \in \Bbbk^{m\times n}$ e $\mathbf{B} \in \Bbbk^{m\times p}$, se tem

$$\mathcal{R}\big([\mathbf{A}\ \mathbf{B}]\big) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B}).$$

1.6. Seja $\mathbf{A} \in \mathbb{C}^{m\times n}$. Prove que:

(a) $r(\mathbf{A}^*\mathbf{A}) = r(\mathbf{A}) = r(\mathbf{A}\mathbf{A}^*)$.

(b) $\mathcal{R}(\mathbf{A}^*\mathbf{A}) = \mathcal{R}(\mathbf{A}^*)$ e $\mathcal{R}(\mathbf{A}\mathbf{A}^*) = \mathcal{R}(\mathbf{A})$.

(c) $\mathcal{N}(\mathbf{A}^*\mathbf{A}) = \mathcal{N}(\mathbf{A})$ e $\mathcal{N}(\mathbf{A}\mathbf{A}^*) = \mathcal{N}(\mathbf{A}^*)$.

Re-escreva as propriedades análogas no caso em que $\mathbf{A} \in \mathbb{R}^{m\times n}$.

1.7. Seja $\mathbf{A}, \mathbf{B} \in \Bbbk^{n\times n}$ matrizes quadradas. Prove que

$$\max\{n(\mathbf{A}), n(\mathbf{B})\} \leqslant n(\mathbf{A}\mathbf{B}) \leqslant n(\mathbf{A}) + n(\mathbf{B}).$$

1.8. Seja $\mathbf{A} \in \mathbb{k}^{m\times n}$ e $\mathbf{B} \in \mathbb{k}^{n\times n}$. Prove que:

(a) Se $r(\mathbf{B}) = n$, então $r(\mathbf{AB}) = r(\mathbf{A})$ e $\mathcal{R}(\mathbf{AB}) = \mathcal{R}(\mathbf{A})$.

(b) Se $r(\mathbf{A}) = n$, então $r(\mathbf{AB}) = r(\mathbf{B})$ e $\mathcal{N}(\mathbf{AB}) = \mathcal{N}(\mathbf{A})$.