Cosmologia Física

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The Homogeneous Universe

Fundamental properties of the zeroth order Universe

Olbers' paradox - Luminosity distance - Universal redshift

Hypothesis: the fact that the night sky is dark may indicate that the Universe is not static.

Let us see why this is so.

Some definitions: Luminosity, Flux (L that reaches the observer), Surface Brightness (Flux concentration)

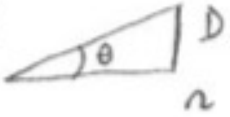
Luminosity $L = E/t$

Flux Luminosity that "reaches us" $F = \frac{L}{4\pi r^2}$ (magnitudes)

Surface brightness S (brilho) Flux per emission area $S = \frac{F}{\Omega}$

Solid angle (2d aperture) $\Omega = \theta^2$

Angular diameter (1d aperture)



$\theta \sim tg\,\theta = \frac{D}{r}$

$$\Rightarrow S = \frac{L/r^2}{(D/r)^2} \sim \frac{L}{D^2}$$

An object of a given Luminosity and Size has a fixed S independent of its distance r.

Surface brightness is the ratio between 2 “apparent” quantities (flux - the apparent magnitude - and angular size - apparent size -) → **the brightness of an object is independent of its distance.**

Two objects of the same intrinsic size and with the same luminosity have the same surface brightness, regardless of their distances from the observer.

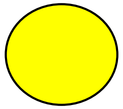
This fact has a very important consequence:

Let us consider two regions of the sky with a given angular size Ω that are completely filled with stars of equal luminosity and intrinsic sizes.

The fluxes of the regions are:

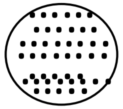
- region A with 1 large object that fills all the region (e.g. the Sun)

$F_A = S \times \Omega$



- region B filled with n stars identical to the Sun

$F_B = S \times \Omega_1 + S \times \Omega_2 + \ldots + S \times \Omega_n = S \times \Omega$



$$\sum_{i=1}^{n} \Omega\_i = \Omega$$

→ **the fluxes from the two regions are equal**.

Naturally, in region B, distant objects have a small angular size, but looking up to a faint magnitude limit (large distances) we can get an angular density of sources large enough to cover the full aperture.

Conclusion (for stars): If a sky aperture of the same size of the solar disk is filled with stars of luminosities similar to that of the Sun, the flux from that aperture is identical to the one coming from the Sun → **the sky should be always bright (day and night).**

The fact that this does not happen is known as the Olbers' paradox (1823)

Note that in fact this indeed happens for the observed Milky Way stars. In dense regions where stars "fill the regions", the "stellar sky" is bright.
There is no paradox here.

The eyes do not integrate for enough time and cannot detect the flux from faint stars, so most regions are not completely filled and the detected flux from them is lower to the naked eye.
But telescopes can saturate → the sky seen by a telescope is really bright!

However there is still a Olbers' paradox, but it applies only to **cosmologically distant objects** (like distant galaxies), so it applies to the "cosmological sky".
In that case, it is observed that even with an "infinite" integration time, the cosmological sky does not saturate, and this has implications for our modeling of the Universe. Let us see this in more detail.

The brightness of the sky can be computed in a more rigorous way:

Considerer the flux function: dN/dF, the number of objects per flux interval.
(Note that this type of functions - number counts per interval of a certain astronomical quantity - are very used in astrophysics: mass function, luminosity function, etc.)

Consider n galaxies per unit volume

How many galaxies have fluxes in the range F to F+dF

Radial shell

$$V_{shell} = 4\pi r^2 dr$$

Number of stars in the shell $dN = n\, 4\pi r^2 dr$

Now, $F_{1\,gal} = \frac{L}{4\pi r^2} \Rightarrow \frac{dF}{dr} = -\frac{L r^{-3}}{4\pi}$

(assume Luminosity is constant, i.e., equal in all objects of the sample, and non-evolving, no r dependence)

Number of galaxies with flux between F and F+dF is $dN = \frac{dN}{dF} dF$ (definition)

$$dN = \frac{dN}{dF} dF = \frac{dN}{dr} \frac{dr}{dF} dF = \frac{n}{L} 4\pi r^2 \, 4\pi r^3 dF \propto r^5 dF$$

(there is a one-to-one relation between flux and distance)

Note that

$$r = \sqrt{\frac{L}{4\pi F}} \propto F^{-1/2} \quad \Rightarrow \quad \boxed{dN \propto F^{-5/2} dF}$$ flux function

There are more objects with small flux - the distant ones - than with large flux - the closer ones - (at equal luminosity)

Integrate up to the detection flux-limit (magnitude limit):

Now we can compute the total flux of stars brighter than a certain minimum F_0 :

$$F_{\rm tot} = -\int_{F_{\rm min}}^{F_{\rm max}} F \frac{dN}{dF} dF \sim -\int_{F_{\rm min}}^{F_{\rm max}} F^{-3/2} dF \sim F_{\rm min}^{-1/2} - F_{\rm max}^{-1/2} \to \infty$$

(when F_min → 0)

(this is the standard way to compute a total or a weighted mean - the flux function is a weight function)

If we get the flux of objects up to F ~ 0 (i.e., including objects up to r → ∞), then the total flux would be infinite → the bright night sky

Why is the (cosmological) night sky not bright ?

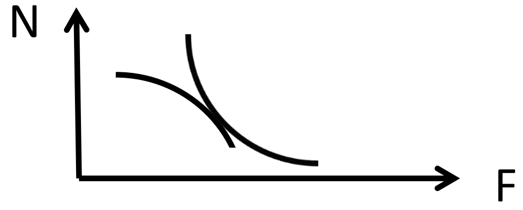
In reality we cannot integrate up to infinite distance (flux zero) if the object is not eternal (has an initial time). There is a cut-off F_min > 0 and the integral is finite. However it could still be very large → **Assuming an initial time does not solve the paradox.**

Perhaps there is absorption and part of the flux is lost?
True, but there would be re-emission of the absorbed flux that would still contribute to the total flux, even if in another form (such as with a different wavelength) →
Absorption is also not the solution.

Possible solution of the paradox:

To solve the problem in an absolute way, the best would be to obtain a total flux that would not go to infinity even in an infinite universe (i.e., even in the case F_min → 0). In that case it would be understandable that the night sky was not bright.

This could be achieved if the function dN/dF would be different, in particular if it would have a shallower slope → if the number of objects with small flux was smaller than predicted.



But their number on each spherical shell must increase with r^2 in a scenario of uniform distribution (homogeneity).

Moreover, even if some objects would disappear (end of life), others would appear to replace them (and why would this affect more the distant than the closer objects?) → **Finite life-time is also not a solution.**

However, what if the distant galaxies would contribute less to the flux? → meaning, they would have a smaller brightness → i.e., it would be like a smaller effective number of galaxies (even though the number would not change). But we saw that brightness does not depend on distance … or does it?

Could S become distance-dependent?

i) A possibility would be if Luminosity L or size D were distance-dependent → **all objects would evolve in time** (since the more distant ones are in the past) **in a universal way**, such that luminosity would always increase (smaller in the past) - or the intrinsic size would decrease (larger in the past) → **universal intrinsic evolution of luminosities or sizes.**

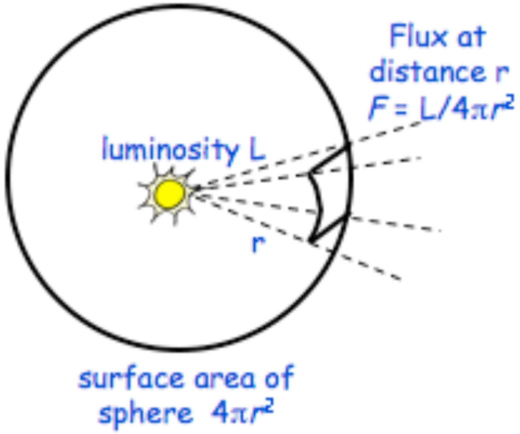
It seems unlikely to happen! and in fact this is not observed

ii) Another possibility would be that the flux (the numerator in the expression for S) does not change with r^2, but with a different f(r). This could happen if there exists a mechanism that would make the luminosity emitted by the distant objects to be somehow diluted *during propagation* → **universal loss of luminosity.**

Note that this is different than the first possibility, where the intrinsic luminosities of all objects would decrease (an astrophysical evolution).

This loss of luminosity during propagation would need to **increase with distance**, for the effect to go in the right direction.

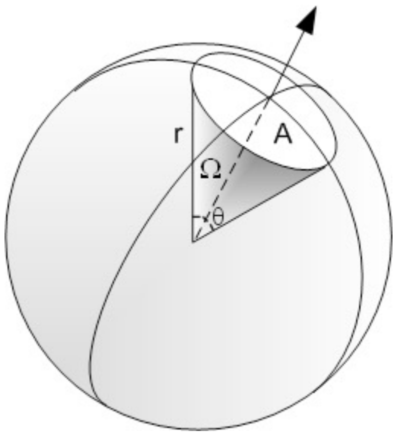
This also seems unlikely to happen! Needs to be tested with observations!



In other words, the hypothesis is that

flux(r) at a distance r from the source is less than L/r^2

while the angular size of a source of intrinsic area D^2 is the usual D^2/r^2



$$S = \frac{L/\overset{d_L}{\boxed{r}}^{2}}{\left(D/\underset{d_A}{\boxed{r}}\right)^{2}}$$

The angular size / intrinsic size relation would be the true geometrical distance ‘r’ → the ”angular diameter distance” d_A

The flux / luminosity relation would depend not only on the geometrical distance but also on an extra factor of “luminosity loss” → by convention, this factor is absorbed in an effective ‘r’ in the numerator, defining an effective distance different from ‘r’ → the “luminosity distance” d_L

For this mechanism to solve the paradox the two distances must be related as

$d_L = f(r)\, d_A$ (i.e., the extra factor must be function of ‘r’).

What mechanism could produce this effect?

Hypothesis: a universal change in all photons wavelength as they propagate from source (e) to observer (o) can produce this effect.

In particular, we need a **redshift** (not a blueshift), because the goal is to decrease the contribution of distant sources (not nearby ones).

Redshift is defined as $z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{\Delta t_o - \Delta t_e}{\Delta t_e}$ (if $z < 0 \Rightarrow$ blueshift)

The existence of a redshift modifies the luminosity propagation in two ways:

- decrease of frequency → universal increase of the time interval between two pulses → $\Delta t_0 = \Delta t_e (1+z)$ → less photons arriving per unit of time

- decrease of frequency (increase of the photons' wavelength) → universal loss of energy $E_0 = E_e /(1+z)$

Remember that

$$L_0 = \frac{E_0}{\Delta t_0}$$

and so the combination of the two effects creates a **luminosity loss of $(1+z)^2$**

By convention, this factor is absorbed in the definition of a new distance → the "luminosity distance" d_L

$$\mathbf{d_L = (1+z(r))^2 \, d_A}$$

This relation is known as **Etherington's distance-duality relation**

Measurements of d_A and d_L are used to test this relation at various redshifts.
If a deviation from $(1+z)^2$ is found, it means that the luminosity loss is not caused by redshift (or *only* by redshift), but there are other effects contributing to it:

non-conservation of photon number? → it would be a hint for **new physics**.

(e.g., Martinelli et al 2020, https://arxiv.org/pdf/2007.16153.pdf)

Let us now insert the result in the expression for the surface brightness:

$$S = \frac{L}{D^2}\left(\frac{d_A}{d_L}\right)^2 = \frac{L}{D^2}\left(\frac{1}{f(1+z(r))}\right)^2$$

We confirm that the brightness is no longer distance-independent, but becomes redshift-dependent:

$$S = \frac{L}{D^2}\left(\frac{d_A}{d_L}\right)^2 = \frac{L}{D^2}\frac{1}{(1+z)^4}$$

This extra factor of $(1+z)^4$ solves Olbers' paradox, since the flux no longer diverges in the small flux limit:

$$F_{\mathrm{tot}} \sim \int_{F_{\mathrm{min}}}^{F_{\mathrm{max}}} F^2 F^{-5/2} dF \sim F_{\mathrm{max}}^{1/2} - F_{\mathrm{min}}^{1/2}$$

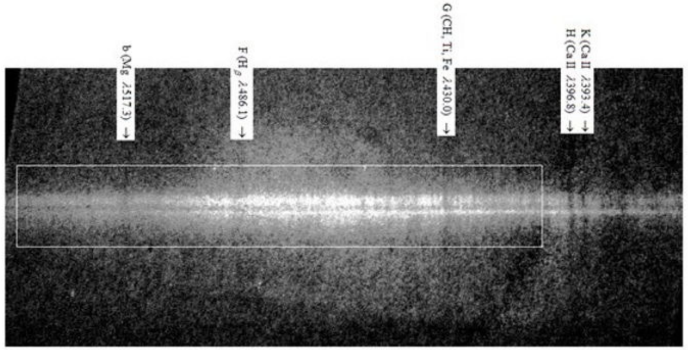
(remember that the earlier result was $F^{-1/2}$)

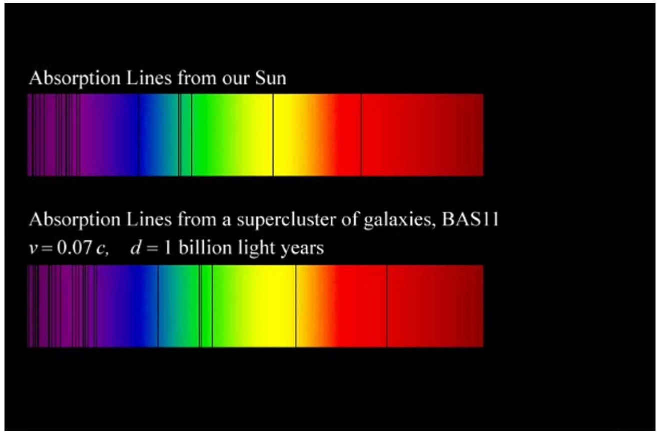
The universal redshift is capable of explaining why the cosmological sky is not bright. Now, is there a plausible mechanism that can produce this type of universal redshift?

Hubble's law - Expansion

Redshift observed: it was observed that all "nebulas" (galaxies) had a redshift.

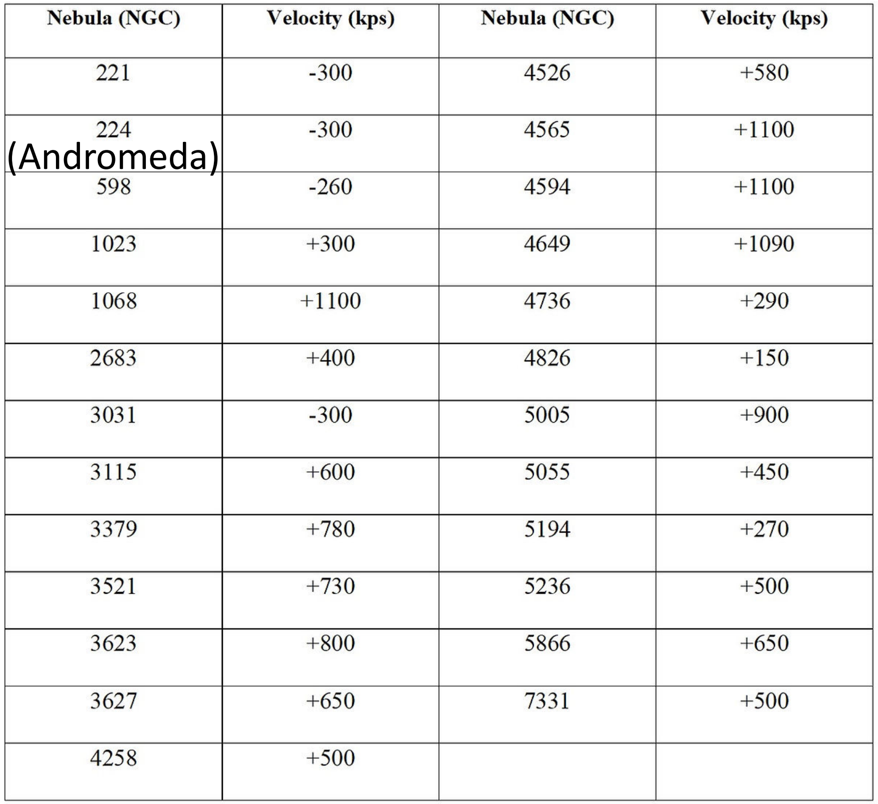
Vesto Slipher (Obs. Lowell)
measured redshifts in galaxy spectra (1912-1922)
41 galaxies, most with $z > 0$





The large velocities were also a first indication that these "nebulas" were outside our galaxy

Milton Humason (Mt. Wilson)
measured redshifts in galaxy spectra (1920)



M31 (Andromeda)

Nebula (NGC)	Velocity (kps)	Nebula (NGC)	Velocity (kps)
221	-300	4526	+580
224	-300	4565	+1100
598	-260	4594	+1100
1023	+300	4649	+1090
1068	+1100	4736	+290
2683	+400	4826	+150
3031	-300	5005	+900
3115	+600	5055	+450
3379	+780	5194	+270
3521	+730	5236	+500
3623	+800	5866	+650
3627	+650	7331	+500
4258	+500		

Edwin Hubble (Mt Wilson)

measured galaxy distances (1923-1926) by identifying the absolute luminosity of the brightest stars.

This is a method with very low accuracy.

Georges Lemaître combined the 2 types of measurements and found a linear relation

v = constant . d (1927)

with a slope H_0 = 625 Km/s/Mpc

[Ann. Soc. Sci. Brux.]

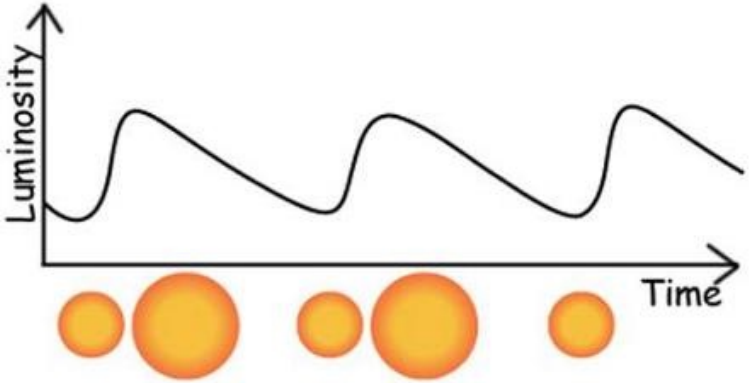
Universal redshift observed: it was realized that the redshift of all observed galaxies increased with their distance (linearly).

The observations support the expanding model that had already been proposed as possible solutions of Einstein equations: (**Friedmann** 1922)

Hubble and Humason

new distance measurements using Cepheids (1927-1929)

The measurements were made by identifying Cepheid stars on the observed galaxies. These are variable stars (pulsating radially).



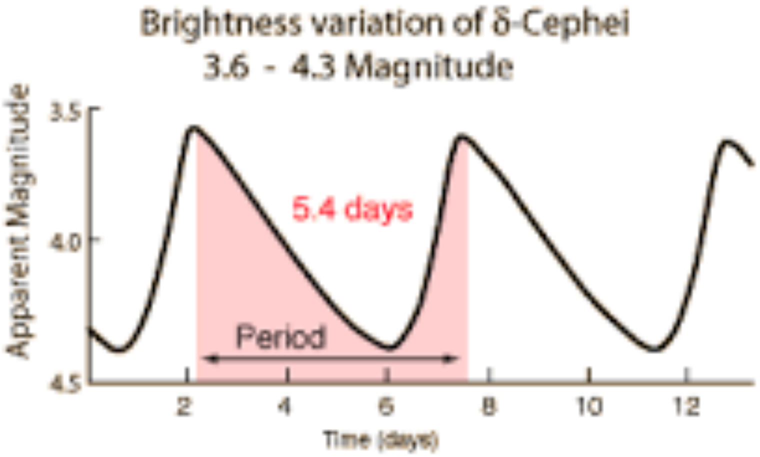
They are very bright stars (10 000 times more than the Sun).

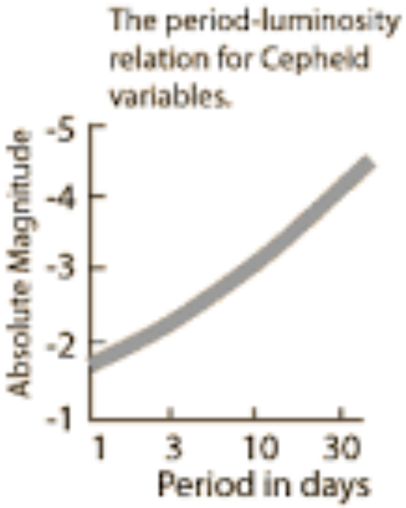
They are in a thermodynamical unstable state: they have a layer of compressed ionized He at a depth such that it traps heat produced by the star making the star to expand. The expansions makes it to cool down and He recombines (neutral gas) => energy escapes and the star falls inward, compressing He again → a cycle is produced.

The main point is that the Period of oscillation is related with the luminosity.

$\uparrow P \Rightarrow \uparrow L \Rightarrow \downarrow M$

P → L
L + F → D





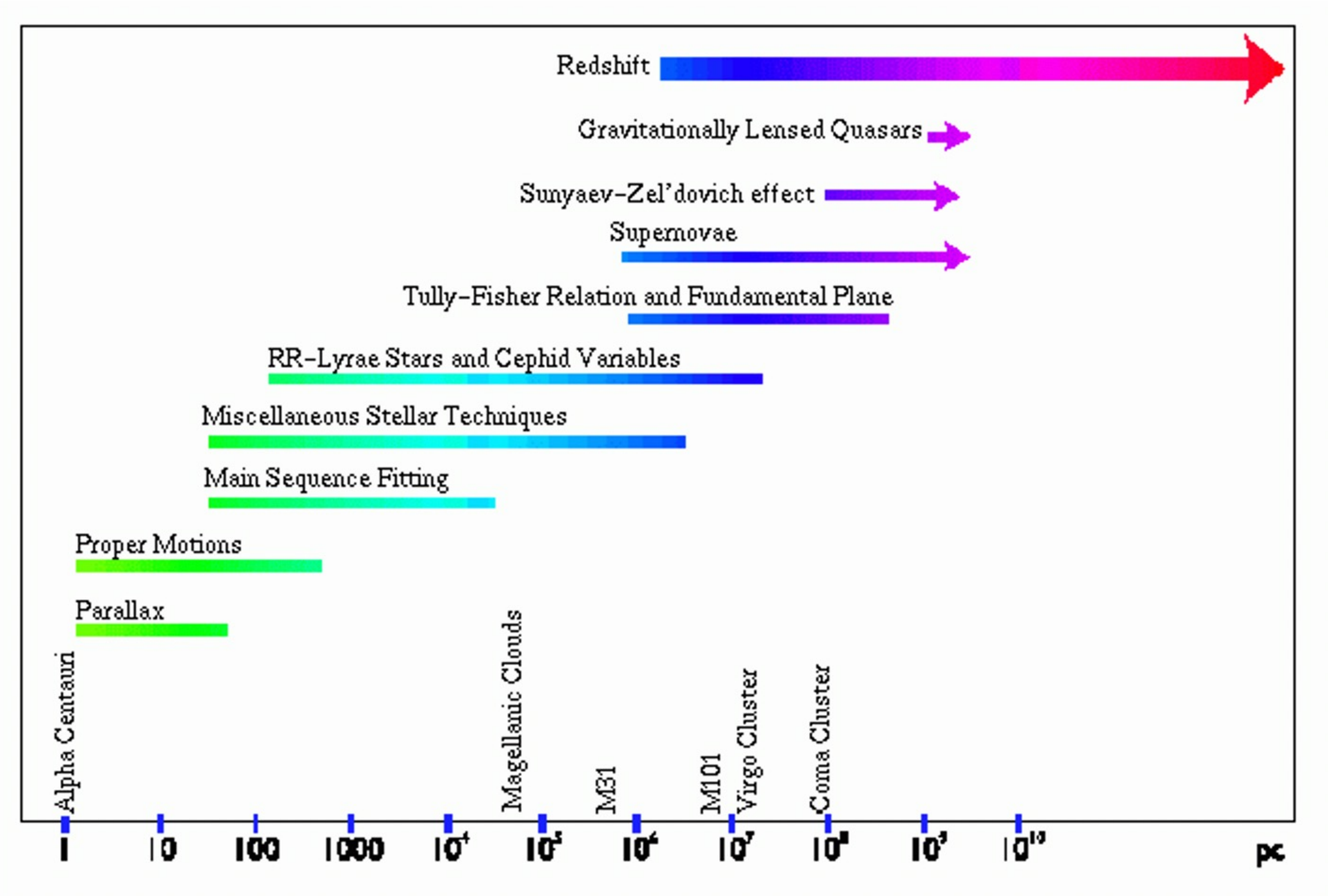
The period increses with the luminosity peak.

So, the period is the proxy for the distance (subject to calibration)

The absolute values of the period-luminosity relation are calibrated with observations (more reliable than to calibrate from a theoretical model for the astrophysics of these stars) → need to observe other Cepheids with known distances (D + F → L). Those are Cepheids in our galaxy (eg: polaris or δ Ceph)

The distances to these nearby Cepheids are obtained by parallax.
(Earth-Moon eclipse, Earth-Sun baseline, 1pc = 1arcsec)

These are the first steps of the so-called distance ladder.



Universal redshift confirmed

Hubble combined the redshift measurements with the new distance measurements and found a linear relation v(d) (1929) with a slope H_0 = 530 Km/s/Mpc

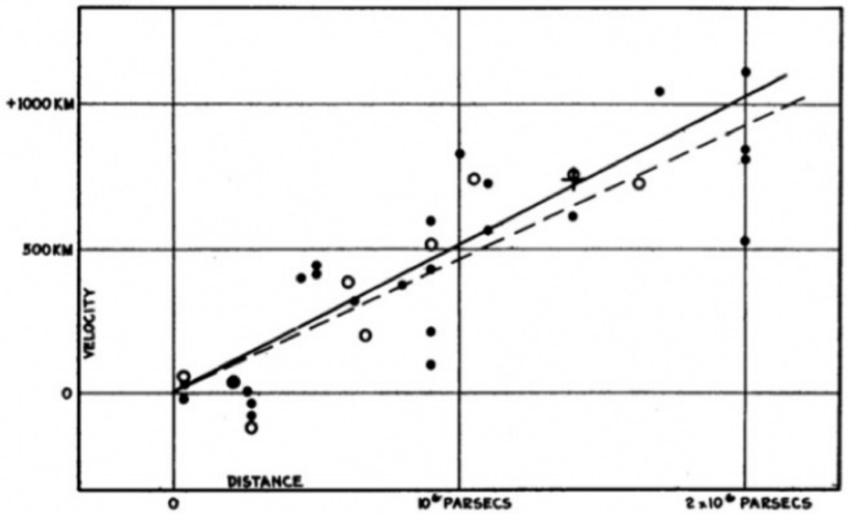


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

This correlation was interpreted as a universal recession of the galaxies because the redshift was interpreted as a velocity through Doppler’s effect:

v/c = z

Lemaître published the english translation of his paper in MNRAS (1931), but his proposal of a linear correlation and his H_0 value were not included, and his role in the discovery was lost.

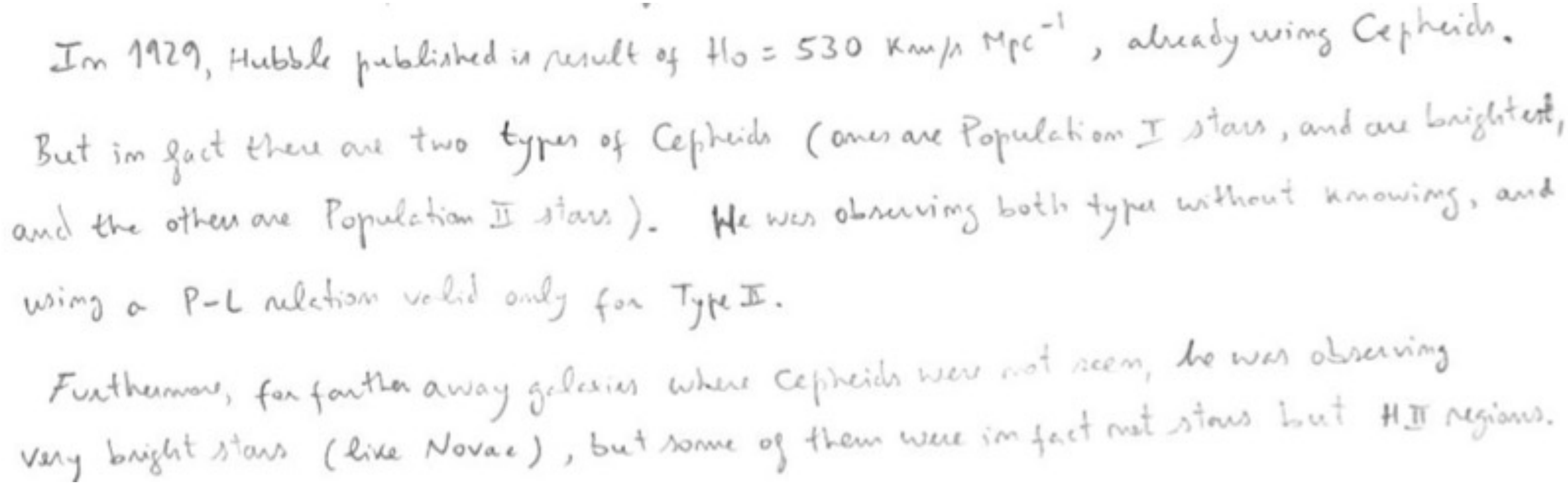
Until recently the reason for the non-inclusion of the main results of Lemaitre in the MNRAS paper was a mystery. **Was it a conspiracy made by Hubble?**

In 2011, Mario Lívio researching the letters between Lemaitre, the translator and the editor, found out that Lemaitre himself has asked to not include the results that he considered were already “old news”.

More recently, this issue was debated in the annual meeting of the IAU (2018) and there was a voting open to the worldwide research community, to propose the change of the naming of Hubble’s law.

78% of the votes approved the change: since N**ovember 2018, Hubble’s law is now named Hubble-Lemaitre’s law.**

We now today that these numbers are too high: H_0 = 530 Km/s/Mpc



In 1929, Hubble published in result of $H_0 = 530$ Km/s Mpc^{-1}, already using Cepheids.

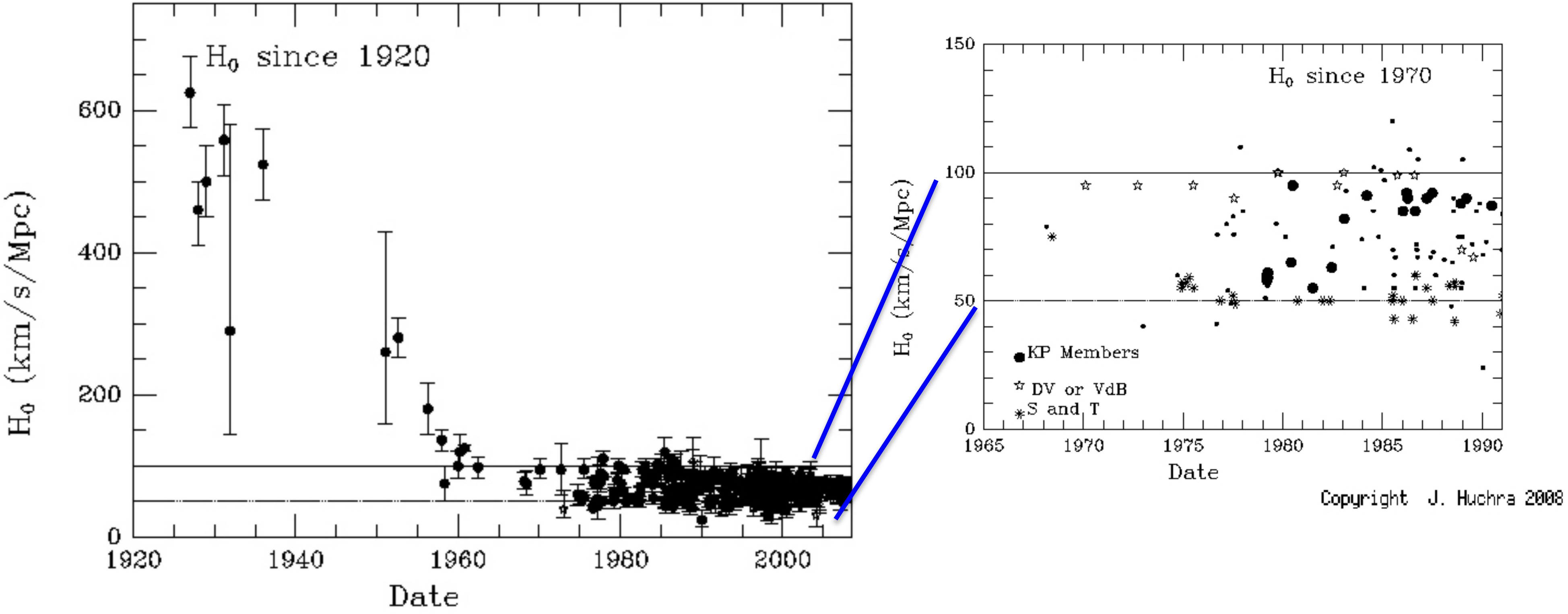
But in fact there are two types of Cepheids (ones are Population I stars, and are brightest, and the others are Population II stars). He was observing both type without knowing, and using a P-L relation valid only for Type II.

Furthermore, for farther away galaxies where Cepheids were not seen, he was observing very bright stars (like Novae), but some of them were in fact not stars but HII regions.

This mistake was identified by W. Baade in the 1950s.

The determination of the Hubble constant has dominated observational cosmology throughout all the XXth century!

only in the 21st century did other cosmological parameters start to be measured with higher precision and using a great variety of methods → CMB, galaxy clustering, BAO, weak lensing, etc. → precision cosmology



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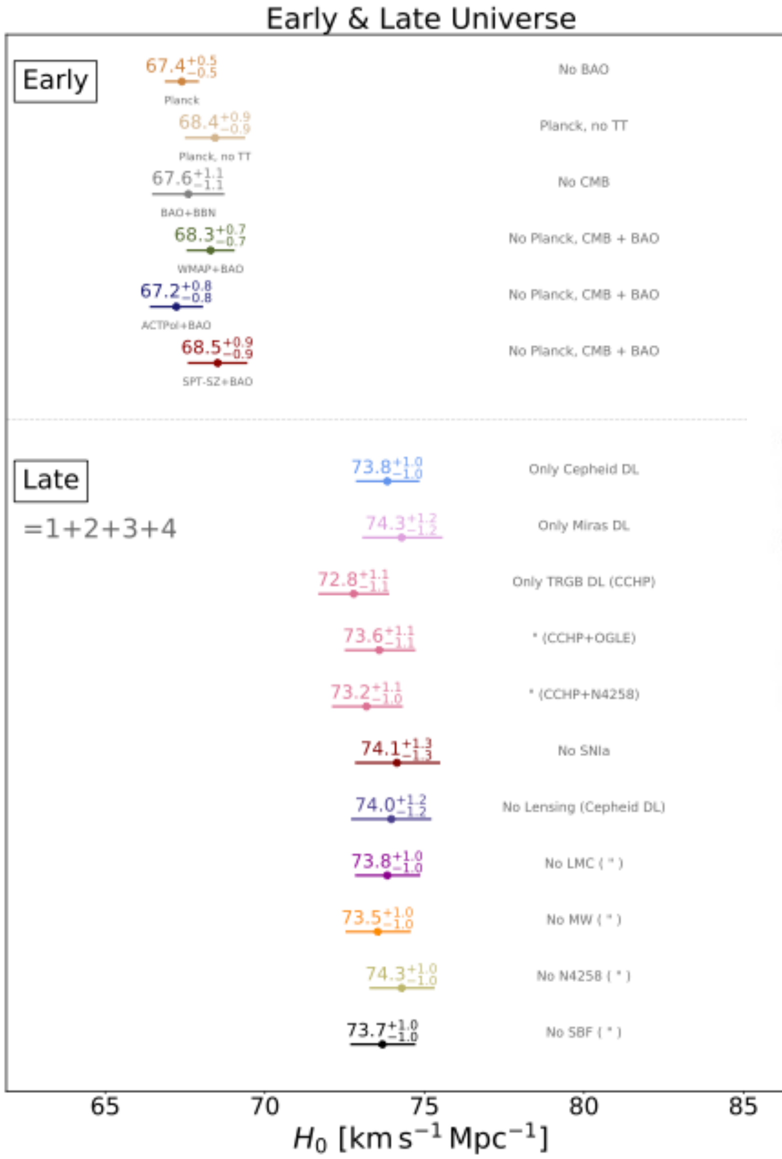
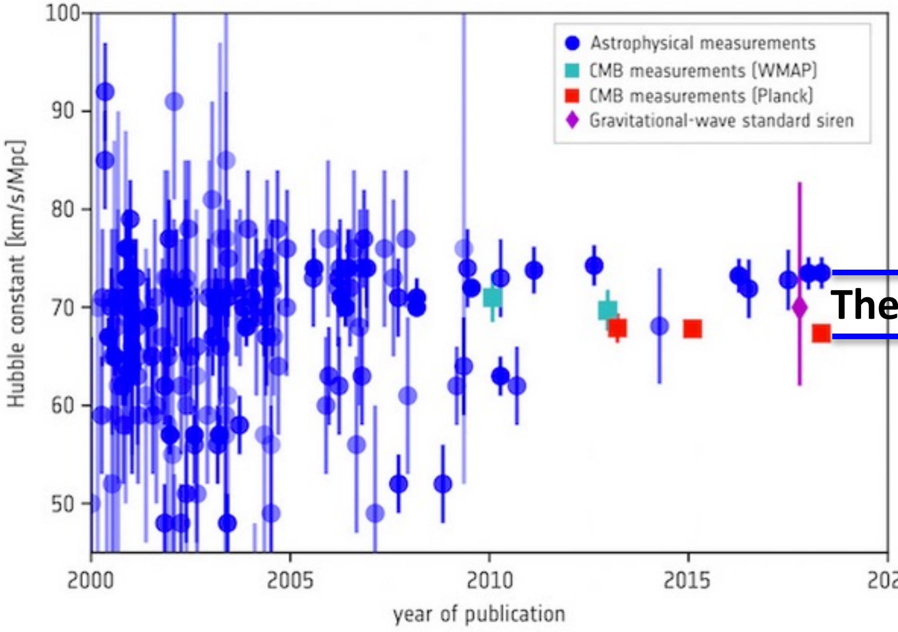
Long discrepancy between 2 groups: h ~1 e h ~0.5 due to issues with evolution and calibration on sources with peculiar velocities.

The polemic only ended in 2001 with the HST Key Project:
by observing Cepheids and supernovae in the same galaxy → h = 0.72 ± 5%

However the debate re-opened in the last decade with the H_0 measurements made by the CMB Planck mission (and other surveys of the early Universe) finding lower values → 0.67 ± 1%



There is a systematic separation between the higher (lower) values found with late (early) universe data.



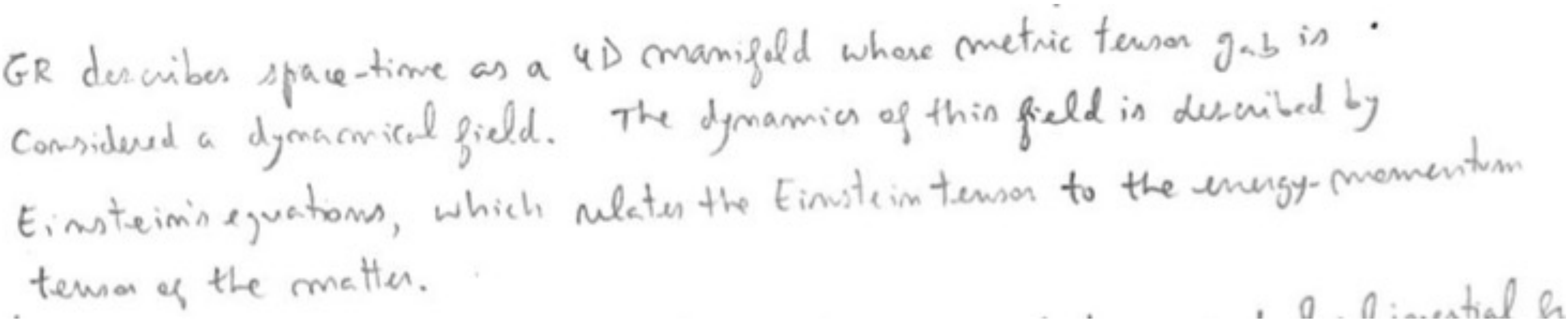
The Hubble tension

Further reading:

Riess et al 2020, https://arxiv.org/pdf 2001.03624.pdf

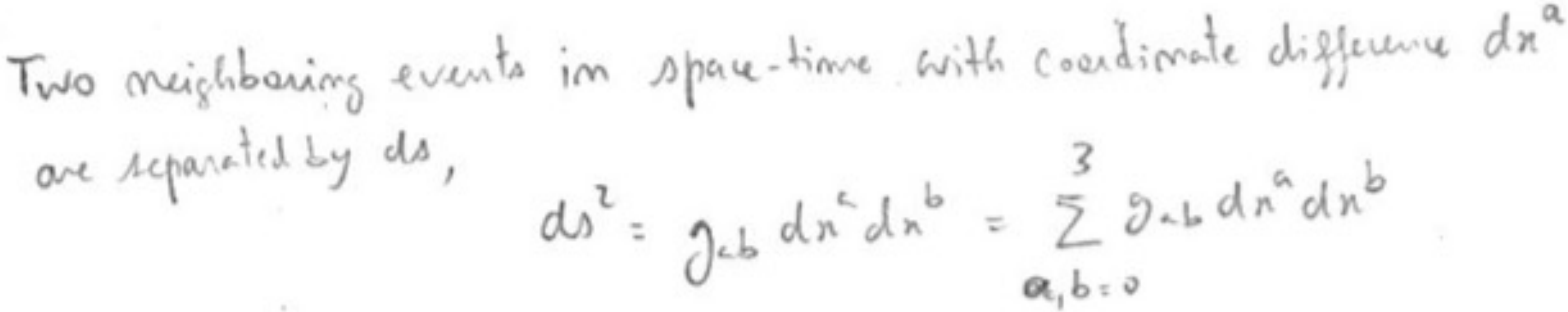
Knox & Millea 2020, https://arxiv.org/pdf/1908.03663.pdf

The metric



GR describes space-time as a 4D manifold where metric tensor g_{ab} is considered a dynamical field. The dynamics of this field is described by Einstein's equations, which relates the Einstein tensor to the energy-momentum tensor of the matter.

The metric keeps its traditional role of determining distances and local inertial frames, but it also plays the dynamic role of a gravitational potential, determining the geodesics (trajectories) of the space-time.



Two neighboring events in space-time with coordinate difference dx^a are separated by ds,

$$ds^2 = g_{ab}\,dx^a dx^b = \sum_{a,b=0}^{3} g_{ab}\,dx^a dx^b$$

Let us derive the metric of a homogeneous and isotropic space-time — for observers that follow the mean motion of matter and radiation in the universe (the expansion) - the comoving observers; they have $(r, \theta, \varphi) = \text{constant}$

The most general metric can be written as

$$ds^2 = -g_{00}\,dt^2 + 2g_{0i}\,dx^i dt + \sigma_{ij}\,dx^i dx^j$$

The general form of the metric is fixed by the cosmological principle:

Isotropy implies $g_{0i} = 0 \rightarrow$ otherwise this would define a prefered direction.

(static)

The comoving observers are synchronized (comoving with the fluid)

$\Downarrow$

single time for all can be redefined and remove g_{00}

$$-g_{00}\,dt^2 \longrightarrow dt^2$$

Homogeneity implies there are only functions of $\underline{t}$:

$$ds^2 = -dt^2 + a^2(t)\ d\ell^2(x,y,z)$$

$$\downarrow$$

the time-dependence may be factored out as the scale factor

Homogeneity and isotropy leads to δ_{ij} with spherical symmetry

The general metric that verifies the cosmological principle is the **Robertson-Walker metric.**

Is the metric is completely fixed by the cosmological principle, or are there some **degrees of freedom**?

Scale factor: RW metric first degree of freedom

There is freedom in the time-evolution of the metric, defined by the scale factor a(t).

a(t) is not fixed by the cosmological principle (i.e., by the fact that the metric is RW). It is free to be determined by the dynamics of the Universe (the differential equations provided by GR: the Einstein equations).

Note that in general in a physical system there are several levels of "freedom":

i) **Symmetries** impose general constraints (in this case determine the type of metric) but leave the physical functions f(t,x) free (in this case a(t)).

ii) **Differential equations** provided by the theory (in this case the Einstein equations of GR) are solved to get a solution for f(t,x) (in this case the functional form a(t)). The solution always include integration constants, which implies that the solution function can only be determined up to a constant. → This fixes the "model".

iii) **Initial conditions**, i.e., conditions at the borders of the (t,x) domain, that can be time or spatial, provide the absolute value of a physical function (or its derivatives) at a certain point of its domain (t,x) (in this case $a(0)$, $a(t_0)$, $\dot{a}(t_0)$). The initial conditions may be imposed (in this case $a(0) = 0$, $a(t_0) = 1$), or determined by observations or experiments (in this case, the value of $\dot{a}(t_0)$, i.e., H_0). → This fixes the "cosmology".

The “initial condition” absolute value of a cosmological physical function (or its derivatives) at a certain point of its domain (t,x) is what is called a **cosmological parameter**.

There are also **phenomenological models**. In these cases, the cosmological functions are proposed empirically and do not come from a solution of a differential equation of a theory. In these cases, the functional form of a cosmological function can also be parameterized → This introduces more cosmological parameters, besides the ones strictly related with initial conditions (that would determine the amplitude of the function).

Solution for a(t) (model-dependent)

a(t) can thus be determined in a model-dependent way, i.e., by solving the relevant differential equations, and its amplitude parameterized by an initial condition.

a(t) is usually parameterized using its amplitude today at t_0.

However, differently from most functions, a_0 is not find by observations but it is just fixed by convention: $a_0 = 1$.

This means that all possible a(t) solutions are “distorted”, i.e., are forced to reach the value a=1 today.

This implies that the parameter a_0 is not useful to distinguish the various cases (the various cosmologies).

Various cosmologies are instead distinguished by looking at the slopes with which the functions $a(t)$ reach $a=1$. In other words, the relevant parameter is

å (today) (notation: å is da/dt).

Since $a_0=1$, we have $\mathring{a}_0=H_0$ → the Hubble constant is a free parameter of the model (the parameter related to the initial condition of $a(t)$) and its value (determined from observations) will help in defining the cosmology.

(It is the first cosmological parameter we encounter).

Solution for a(t) (model-independent)

Note that $\mathring{a}_0$ would also be the first parameter in the Taylor expansion of $a(t)$ around t_0

In fact, if we would Taylor expand $a(t)$ around t_0, introducing a potentially infinite number of parameters (the values of all-orders derivatives at t_0), $a(t)$ would be fully described (in the local Universe).

If we could design a way to measure all those parameters individually, we would then reconstruct a(t) with no need for the evolution equations → in a **model-independent way**.

The set of all those parameters - called the **cosmographic parameters** - would contain the same information as the set of differential equations.

There are attempts to do this, and the lower-order parameters are defined:

$$a(t) = a(t_0) + \dot{a}|_{t_0}(t-t_0) + \frac{1}{2}\ddot{a}|_{t_0}(t-t_0)^2 + \sigma(t-t_0)^3$$

$$\Rightarrow \boxed{a(t) = \left[1 + H_0(t-t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t_0}(t-t_0)^2 + \dots\right] a(t_0)} \quad (a(t_0)=1)$$

The first-order term is a **velocity** term → the **Hubble constant** H_0

The second-order term is an **acceleration** term → the **deceleration parameter** q_0

(historically defined with a minus sign, hence the name deceleration instead of acceleration)

$$q_0 = -\frac{\ddot{a}}{a}\Big|_{t_0}\frac{1}{H_0^2}$$

We can continue the expansion to higher-orders, defining a series of parameters:

$$p_0^{(m)} = \frac{d^m a}{dt^m}\frac{1}{a}\Big|_{t_0}\frac{1}{H_0^m}$$

obtaining

$$a(t) = a_0\left[1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \frac{1}{3!} j_0 H_0^3 (t-t_0)^3 + \frac{1}{4!} s_0 H_0^4 (t-t_0)^4 + \sigma(t-t_0)^5\right]$$

The next parameter is called the **jerk** j_0 (that corresponds to the change of acceleration, well-known in mechanics, felt for example when changing gears in a car)

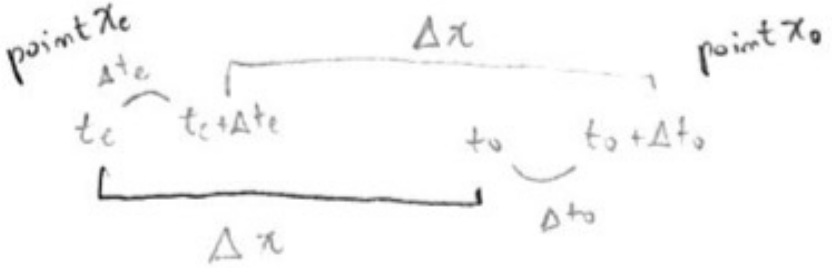
The next orders parameters are called **snap** s_0, **crackle** c_0 and **pop** p_0 (taken from the names of the characters in these cereals!).

This approach to cosmology is called **cosmography**: the Universe is described in terms of **its dynamical quantities** with no need to solve the evolution equations (the Einstein equations) and no need define the sources of gravity (in the energy-momentum tensor).

Expansion - redshift

The universal redshift is caused by the expansion. So, we must be able to derive a relation between the redshift and the expansion degree-of-freedom: i.e a function z(a)

Let us consider the emission of a lightwave (2 pulses) in the expanding Universe:



x_e, x_0 comoving coordinates

Δt_e and Δt_0 may be different.

(due to the expansion)

the 2 pulses around Δt_e define the frequency of the emitted signal
The " " Δt_0 " " — received "

$$\nu_e = \frac{1}{\Delta t_e}, \quad \nu_0 = \frac{1}{\Delta t_0}$$

$$\lambda_e = c\Delta t_e, \quad \lambda_0 = c\Delta t_0$$

Redshift is defined as $z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\Delta t_0 - \Delta t_e}{\Delta t_e}$ (if $z<0 \Rightarrow$ blueshift)

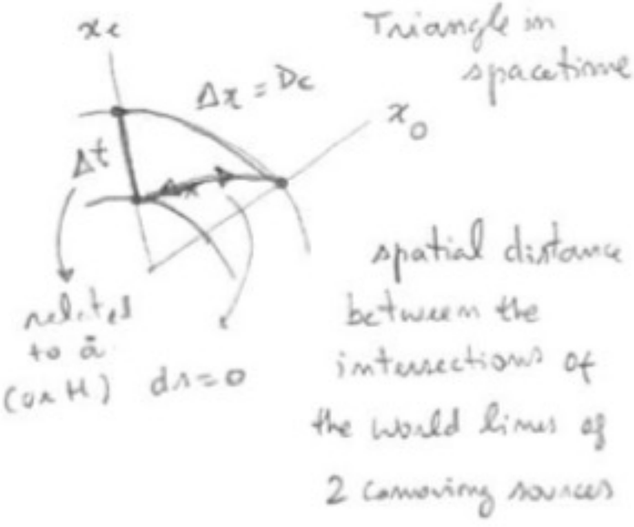
Since we are dealing with non-instantaneous light propagation, we will need to use (special) relativity (no need for GR, no dynamics involved) (flatness is assumed).

Let us compute Δt_e and Δt_0 using the metric:

$$ds^2 = -dt^2 + a^2\left(d\chi^2 + f_K^2(\chi)\,d\Omega^2\right)$$

Note that this relation is space-time trigonometry!

$ds^2 = 0$ null geodesic $\rightarrow$ $dt = a\,d\chi$



Knowing the “hypotenuse”(ds^2=0) and one side of the triangle (dx^2), we get the other side (dt^2).

$$\Delta\chi = \int_{\chi_e}^{\chi_0} d\chi = \int_{t_e}^{t_0} \frac{dt}{a(t)} = \text{Constant}$$

The “space-time triangle” allows us to find a relation between the time ratios and the scale factor:

$$\Delta x = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} \longrightarrow = \int_{t_e+\Delta t_e}^{t_0} \frac{dt}{a} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a} =$$

$$= \int_{t_e}^{t_0} \frac{dt}{a} \underbrace{- \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a}}_{\text{assume } a(t) \text{ is cte over this interval}} = \Delta x - \frac{\Delta t_e}{a(t_e)} + \frac{\Delta t_0}{a(t_0)}$$

$$\Rightarrow \boxed{\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)}}$$

$$\rightarrow z = \frac{\Delta t_0}{\Delta t_e} - 1 \quad \Longrightarrow \quad \boxed{z = \frac{a(t_0)}{a(t_e)} - 1}$$

(definição)

with $a(t_0) = 1$, today

$$\Rightarrow z = \frac{1}{a} - 1 \Rightarrow \boxed{a = \frac{1}{1+z}}$$

This is the result: $1 + z(r) = 1 / a(r)$

The model for the expansion is characterized by $a(r)$, where a varies from
$a(r=\infty) = 0$ to $a(r=0) = 1$,

and thus the expansion indeed creates a universal redshift with the required properties → z increases with r, and $z(r=0)=0$

(Note: the monotonic behavior z(a) is the reason why the redshift can be used as a time variable in the evolution of the Universe)

$a(r)$ is a central quantity that characterizes the cosmological model at the homogeneous level. It is determined from the equations of the theory of which the expansion model is a solution (a theory of gravity). Similarly, the behaviours of $a(t)$, $z(r)$, or inversely $r(z)$, should all be predictions of the theory.

Measurements of these functions, especially $r(z)$, i.e. $d_L(z)$ and $d_A(z)$, are widely used to test the cosmological model.

Note that the **Hubble law is also a relation between the scale factor and redshift** (and so it is a direct solution of Olbers' paradox):

Hubble law:

$$v = H_0 r \Leftrightarrow z = \frac{H_0}{c} r$$ (assuming Doppler effect z = v/c)

Now, the observations were made at a = 1 and are valid for the local Universe.

In the local Universe (a ~1) we have 1-a = Δa = Δt (da/dt)_t_0 (Taylor expansion)

→ $$\frac{1-a}{\Delta t} = \frac{\dot{a}}{a}$$ (using $a(t_0) = 1$)

$$H_0 \Delta t = 1 - a$$

Considering Δt = r/c, we get : z = 1- a

→ **so Hubble law tells us that z = 1 - a**

This means that the assumption of the Doppler effect, plus that the linear relation z vs r translates into a linear relation v vs r → implies a linear relation z vs a.

This is not the expression we found before.
But note it is a linear approximation to our expression (Taylor expansion) :

$$a = \frac{1}{1+z} = 1 - z + \mathcal{O}(z^2)$$

This means that only in the local Universe ('a' close to 1, z close to 0) can the redshift be interpreted as a Doppler effect and is the relation redshift vs scale factor (or redshift vs distance) linear.

This relation - **(local) Hubble law -** was the one observed by Hubble.

In general, the relation between redshift and scale factor is not linear **: v = H(t) r**

and the interpretation of the redshift as a Doppler effect would lead to an inconsistency.

Hubble length

We can define a **Hubble length**, since the inverse of the Hubble function has dimensions of time (or length considering c = 1 dimensionless):

r_H (t) - Hubble radius = c/H

Hubble radius today is = 3000 Mpc/h

At $r = r_H$ → $v = c$ → Beyond the Hubble radius, recession velocities are larger than the speed of light. (This is not a problem since the interpretation of the recession as a Doppler efffect is only valid in the local Universe).

Note however that

since a(t) grows → H(t) decreases in time → $r_H(t)$ grows
(e.g. power law expansion $a(t) \sim t^n$ → å(t) $\sim t^{n-1}$ → $r_H(t) \sim t$)

if a(t) grows decelerating (slope $0 < n < 1$) → **$r_H(t)$ grows faster than a(t)**

This explains an apparent paradox: Why in a decelerating Universe **we can observe objects beyond the Hubble radius**, i.e., we detect light coming from points with "recession velocity" larger than c:

Those photons start by being dragged away by the expansion and their proper distance to the observer initially increases. But due to the increase of r_H, those points even though further away have a decreasing recession velocity and they end up being caught by the growing Hubble radius reaching regions where v<c. From that point on, their proper distance starts to decrease, until reaching the observer.

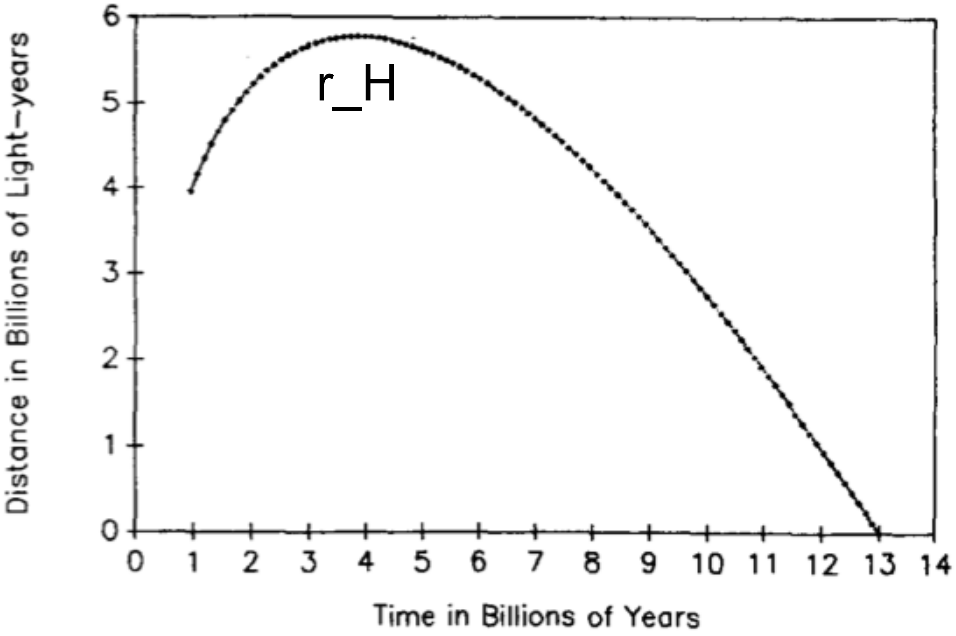


Fig. 1. The photon is emitted from a quasar at $t_e = 0.95$ Gyrs. The quasar (and thus the photon) is 3.8 Gcyrs away from us at time of emission. Initially the photon is "dragged away from us" by the cosmological expansion to a distance of 5.8 Gcyrs at $t = 3.9$ Gyrs. At this time gravity has slowed the expansion rate of space such that the photon is at a coordinate position with recessional velocity of c. The photon then begins to approach us and arrives at $t_0 = 13$ Gyrs.

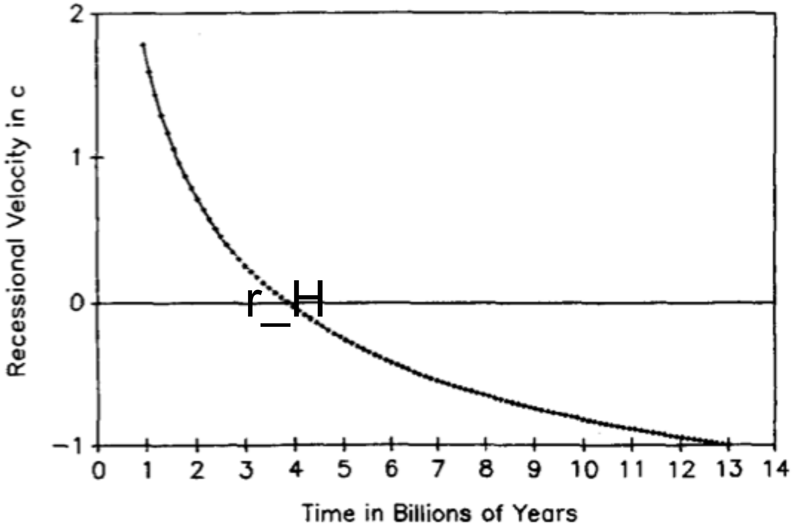


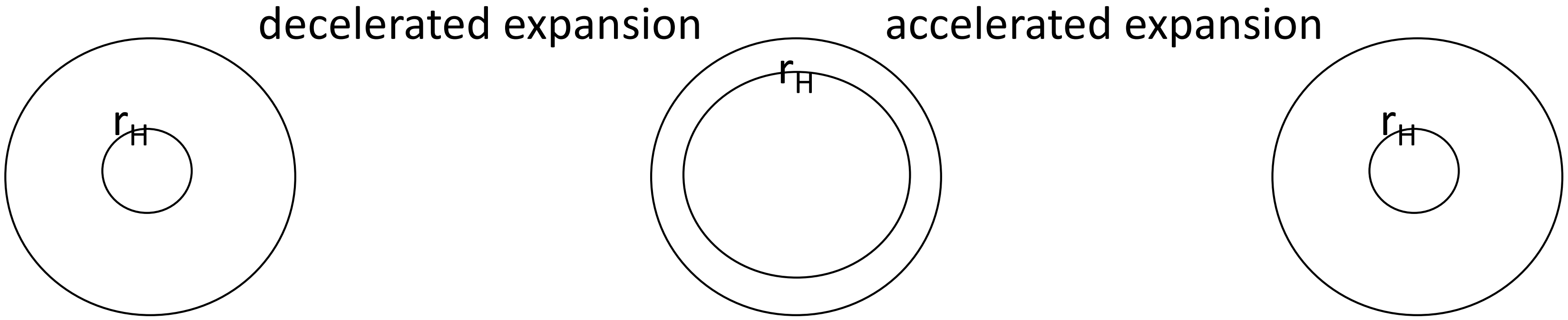
Fig. 2. Because the photon is being "dragged away from us" initially, its recessional velocity is initially positive. At emission the photon is receding at $1.8c$. $\dot{r}_p = 0$ corresponds to the time when gravity has slowed the recessional velocity of space at the photon's position to c. After $t = 3.9$ Gyrs, the amount of expanding space between the photon and receiver is decreasing and the photon approaches the receiver at an increasing rate until the recessional velocity at reception is $-c$.

The fact that the **Hubble radius is not comoving** with the expansion provides a natural way of introducing a feature (a scale) in the homogeneous Universe → r_H is a purely kinematical characteristic scale.

This means that the purely homogeneous Universe at different times is more than just an expanded version of itself.

(purely homogeneous means a completely empty homogeneous Universe, with no structure evolution, or thermal evolution, or radiation emission and peculiar movements, which of course are physical characteristics that allow us to infer there is an evolution).

Picture in comoving coordinates:



decelerated expansion

accelerated expansion

early Universe

late Universe

future Universe

The Hubble radius also provides the weak/strong gravity threshold

Consider **a sphere with Hubble radius**.

Its potential is $V = G\,M / r = G\,4/3\,\pi\,\rho\,r^{2}$:

$GM/r_H = 6.67 \times 10^{-11}\ 4/3\ \pi\ 9 \times 10^{-27}\ (3000/0.7)^2\ (3.1 \times 10^{22})^2$ N m/Kg (i.e. m^2/s^2)

using $G = 6.67 \times 10^{-11}$ N m^2 / Kg^2

$\rho = 9 \times 10^{-27}$ Kg / m^3

$r_H = 3000/0.7$ Mpc

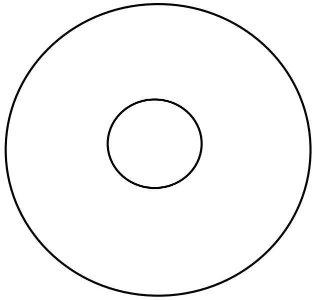
1 pc = 3.1×10^{16} m

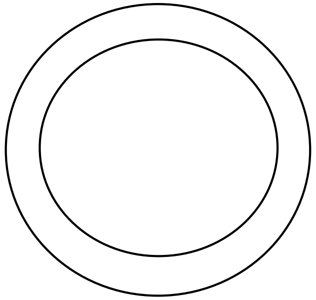
This means that $GM/r_H = 4.44 \times 10^{16}$ N m / Kg

Note that [N m /kg] = [$(m/s)^2$] → the corresponding velocity is

$v = 2.1 \times 10^5$ Km/s ~ order of c

and so the Hubble radius marks the limit of Newtonian gravity
(also called the “space-time curvature radius of the Universe”, beyond which GR gravity is mandatory and new gravitation effects may arise)


early Universe (more relativistic)


late Universe (more Newtonian)

→ for $r < r_H$ we can use Newtonian mechanics (plus special relativity) and consider a flat space-time,

and indeed **we saw that fundamental concepts such as the cosmological principle, the expanding Universe and the redshift do not arise from general relativity.**

Note also that from $V = G M / r \sim \rho r^2$ we conclude that

→ for the same density, a larger region is a stronger gravitational field (hence "more relativistic") → GR effects are important on very large-scales (Newtonian gravity breaks down)

On the contrary on smaller (and also large, up to the Hubble radius) regions of the Universe, Newtonian gravity is still a good description. This includes galaxies, clusters, and even many large-scale structures (LSS) and filaments.

→ for the same size, a denser region is a stronger gravitational field (hence "more relativistic") → GR effects are important on very dense small-scales (like black holes)

This is another strong gravity regime, the **small-distance limit**, where interactions are strong even with smaller masses. This gives rise to the strong gravity effects for black holes and the PPN corrections in solar system mechanics.

So GR is very relevant for very large scales cosmology and black holes physics, but less relevant in astrophysics.

Curvature: RW metric second degree of freedom

Let us look at the spatial part of the metric that, due to the cosmological principle, must be spherically symmetric.

Usually, spherical symmetry is written like,

$$ds^2 = -dt^2 + a^2(t)\left[\underbrace{dr^2}_{\text{radial displacement}} + \underbrace{r^2}_{\text{radius of the sphere}}\overbrace{(d\theta^2 + \operatorname{sen}^2\theta\, d\varphi^2)}^{d\Omega^2}\right]$$

$$(d\Omega^2 = d\theta^2 + \operatorname{sen}^2\theta\, d\varphi^2)$$

But as we know, gravity in GR is the space-time curvature,

and space itself (not only space-time) may also be curved.

So we need to include the additional degree of freedom of curvature in the metric.

How to do this?

To do this, we consider the following feature of a curved space:

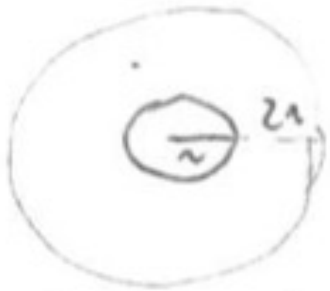
In a curved (3D) space, the area of a spherical surface at distance r from the origin is not $4\pi r^2$ but it is smaller (positive curvature) or larger (negative curvature) than $4\pi r^2$.

So the spatial part of the metric may be written more generally as,

$$dl^2 = u^2(r)\, dr^2 + v^2(r)\, d\Omega^2$$

Only for flat space is $v^2(r) = r^2$:

If the radius is the same as the radial coordinate then,



For positive curvature, we need v(r) < r or u(r) > 1

If $v(r) < r$ => When making a displacement $ds = dr$, we move to a point belonging to a sphere with radius < r

How is this possible geometrically? Introducing curvature!

(Needs higher dimension to picture the curvature)

move r but the radius does not increase proportionally to r.

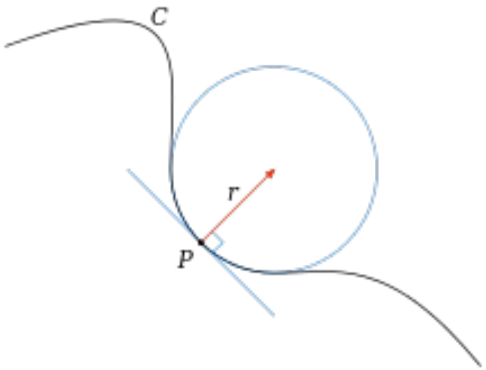
This is <u>positive curvature</u>

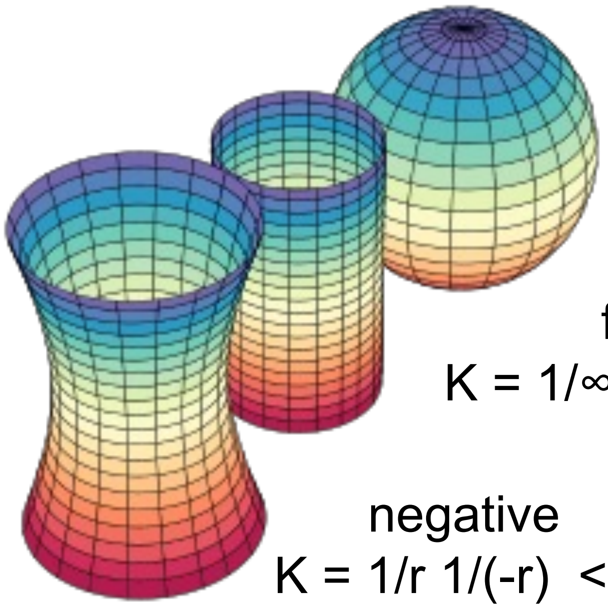
or alternatively if $u(r) > 1$ => dr moves on an effectively larger displacement, but the radius just grows with r.

For negative curvature, we need $v(r) > r$ or $u < 1$

$u>1$ { Radius increases less than the radial displacement is positive curvature
$u<1$ { Radius " more " " is negative curvature

The curvature K of a surface is the product of its **principal curvatures**.
Each principal curvature is $1/r$, where r is the radius of the circle in the normal plane, that best fits the curvature.





positive
$K = 1/r\ 1/r > 0$

flat
$K = 1/\infty\ 1/\infty = 0$

negative
$K = 1/r\ 1/(-r) < 0$

We need now to find out from all possible functions $v(r)$ and $u(r)$ which are the ones that verify spherical symmetry in curved spaces. Or in other words, **what is the constraining condition of a spherical surface**: (in 3D flat space the points on a spherical surface verify the constraint $dx^2 + dy^2 + dz^2 = \text{constant} = dl^2$).

In a curved 3D space we need to consider that the space “curves into a 4D flat space”, i.e., a curved 3D volume may be **embedded** as a 3D-surface in flat 4D space.

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$

The spherical 3D surface is then a constraint on the 4D coordinates. It is simply the surface with points at a fixed radius R from the center of the 4D space:

$$R^2 = \underbrace{x^2 + y^2 + z^2}_{r^2 \longrightarrow \text{(raio 3D)}} + w^2$$

where the combination $x^2+y^2+z^2$ corresponds to the radius of the 3D volume in the 3D space.

So the points in the 4D space such that $\mathbf{w^2 + r^2 = constant}$, are the ones that define the spherical surface.

The line element of that surface is thus $dl^2 = dx^2 + dy^2 + dz^2 + (dw/dr)^2\, dr^2$.

Since the w coordinate of the points on the spherical surface is $w^2 = R^2 - r^2$, we get:

$$w^2 = R^2 - r^2 \Rightarrow 2w\,dw = 0 - 2r\,dr \Rightarrow dw = \frac{-r}{w}\,dr \Rightarrow \left[dw = -r\,\frac{dr}{\sqrt{R^2-r^2}} \right]$$

Then $dl^2 = dx^2 + dy^2 + dz^2 + \dfrac{r^2}{R^2-r^2}\,dr^2$ is the solution

changing to spherical coordinates:

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r\cos\theta \end{cases}$$

$$(d\theta^2 + \sin^2\theta\, d\varphi^2 = d\Omega^2)$$

$$dl^2 = dr^2 + r^2 d\Omega^2 + \frac{r^2}{R^2-r^2}\,dr^2$$

$$\Rightarrow dl^2 = \frac{R^2-r^2+r^2}{R^2-r^2}\,dr^2 + r^2 d\Omega^2$$

$$dl^2 = \frac{1}{1-\frac{r^2}{R^2}}\,dr^2 + r^2 d\Omega^2$$

Note that $1/R^2$ is the curvature of a spherical surface of radius R: K = 1/R 1/R
and so the line element is,

$$dl^2 = \frac{1}{1-Kr^2}dr^2 + r^2 d\Omega^2$$

It corresponds then to a change in the function u(r) , which is $1/(1-Kr^2)$ instead of u(r)=1, while v(r) is kept as r^2

We can also consider that the constant in the condition $w^2 + r^2$ = constant is negative, to allow for the case of negative curvature.

Note that a negative R^2 does not mean an imaginary radius. It simply means that the equivalent radius of one of the principal components is positive and the other is negative (meaning it curves to the opposite side).

This scenario allows for a different solution:

$$-R^2 = r^2 + w^2 \quad \Rightarrow \quad dw = -r\frac{dr}{w} = \frac{-r\,dr}{\sqrt{-R^2-r^2}}$$

Then.

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$

$$dl^2 = dx^2 + dy^2 + dz^2 + \frac{r^2 dr^2}{-R^2 - r^2}$$

$$= dr^2 + r^2 d\Omega^2 + \frac{r^2 dr^2}{-R^2 - r^2}$$

$$= \frac{-R^2}{-R^2 - r^2} dr^2 + r^2 d\Omega^2$$

$$= \frac{1}{1 + \frac{r^2}{R^2}} dr^2 + r^2 d\Omega^2$$

or,

$$dl^2 = \frac{1}{1+Kr^2} dr^2 + r^2 d\Omega^2$$

and now the functions are $u(r) = 1/(1+Kr^2)$ and $v(r) = r^2$

There is thus a degree of freedom associated with the curvature, since there are several possibilities for the curvature of a spherical symmetric space,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{\underset{\searrow\ \text{displacement as function of area}}{1-Kr^2}} + \overset{\nearrow\ \text{Area}}{r^2}\,d\Omega^2\right]$$

that is encapsulated in the curvature parameter K
(K>0: positive curvature; K=0: flat; K<0: negative curvature)

It is also usual to write the metric using the dimensionless curvature parameter k ("small k"):

$$dl^2 = \frac{1}{1-kr^2}\,dr^2 + r^2 d\Omega^2$$

In this case, the curvature types are: **k = 1: positive curvature; k = 0: flat; k = -1: negative curvature.**
Since k is dimensionless, all values of R are equivalent. However in this case, the element dr^2 / $(1-kr^2)$ no longer has dimensions of length, and so the scale factor a(t) needs to have units of length and is no longer dimensionless.

Now, the derivation can also be done in a way that the curvature is encapsulated in the function v(r) instead of in u(r), i.e., u(r) is kept as r, and is v(r) that changes. The result is:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + f_K^2(r)\, d\Omega^2\right]$$

This is the most usual way to write the RW metric

with

$$f_K(\chi) = K^{-1/2}\sin(K^{1/2}\chi)\,, \quad (K > 0)$$

$$f_K(\chi) = \chi\,, \qquad (K = 0)$$

$$f_K(\chi) = (-K)^{-1/2}\sinh[(-K)^{1/2}\chi]\,, \quad (K < 0)$$

Distances

Once we know the metric of the space-time, we can define distances.

In general there is no unique (or correct) definition of distance.

This happens for two reasons:

- the existence of curvature
- the existence of expansion

Effect of curvature

As we saw, the existence of curvature introduces two different radial quantities in the spherical symmetric metric: the radial displacement u(r) and the radius of spherical surfaces v(r).

Both are legitimate ways to define a radial distance.

Looking at the metric,

$$ds^2 = -dt^2 + a^2(t)\,\underbrace{\left(d\chi^2 + f_K^2(\chi)\,d\Omega^2\right)}_{\text{Comoving element}}$$

[note that the comoving radial coordinate (i.e. in the comoving frame not affected by the expansion) is usually written with the letter χ instead of r.]

we can infer that those two distances are Δχ and Δf_K (χ) and they define the:

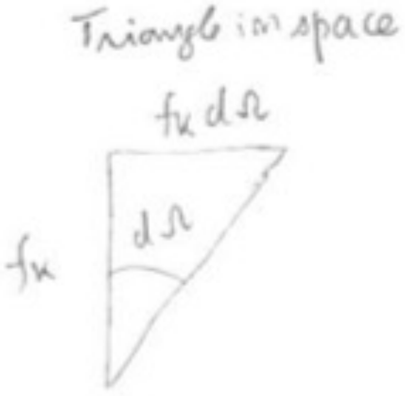
- comoving distance, d_C , also called the **line-of-sight comoving distance** [Δχ]

and

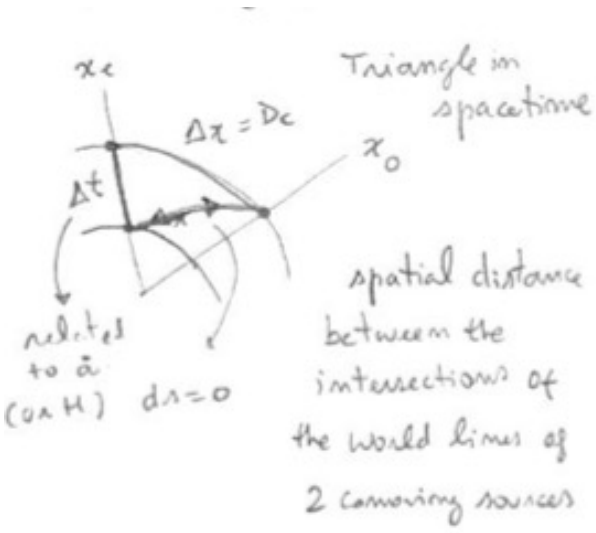
- comoving angular-diameter distance, d_M , also called **transverse comoving distance** or **metric distance** [Δ f_K (χ)]

Since f_K (χ) is the radius of a spherical surface at comoving distance χ from the origin, it is the distance that relates an intrinsic diameter with an angular aperture, hence the name angular diameter distance.

But note that it is a radial distance and not a transverse distance.



To compute these distances from the metric elements, we can consider the **space-time triangle** defined by the line-element (not considering angular variation)



$$ds^2 = -dt^2 + a^2(t)\,d\chi^2 \quad ; \quad ds^2 = 0$$

(the line element is like the theorem of Pythagoras)

$$\Delta\chi = \int_{t_e}^{t_0} \frac{1}{a(t)}\,dt = \int \frac{1}{a}\frac{dt}{da}\,da = \int_{a_e}^{a_0} \frac{1}{a^2}\left(\frac{a}{\dot a}\right) da = \int \frac{1}{H(a)}\frac{1}{a^2}\,da$$

We can also use the redshift as a variable in the integral:

$$z = \frac{1}{a} - 1 \Rightarrow \frac{dz}{dt} = -\frac{\dot a}{a^2}$$

$$\Rightarrow \frac{dz}{da} = -(1+z)^2$$

The resulting expressions for the **comoving distance** are,

$$D_c = \frac{(c)}{H_0} \int_{z_0}^{z} \frac{dz}{E(z)} \quad \text{or} \quad D_c = \frac{(c)}{H_0} \int_{a}^{a_0} \frac{da}{E(a)\, a^2}$$

where the Hubble function is written as $H(z) = H_0\, E(z)$. This is used to separate the functional form and the H_0 parameter value.

The actual value of the comoving distance between two points in the Universe depends on the cosmological model of that Universe. It requires the knowledge of E(z) (obtained from the Einstein equations for the particular model), and of the cosmological parameter H_0 (obtained from observations).

It is usual to absorb the dependence on H_0 into the units of distance, i.e, **cosmological distances are usually given in M_{pc}/h, instead of M_{pc}** (megaparsec), hiding the explicit dependence on the unknown value of H_0

The **comoving angular-diameter distance,** $\Delta f_K(\chi)$, or d_M can be related to d_c using $f_K(\chi)$:

$d_M = \frac{c}{H_0} \sin\left(d_C \frac{H_0}{c}\right)$ positive curvature

$d_M = d_C$ flat

$d_M = \frac{c}{H_0} \sinh\left(d_C \frac{H_0}{c}\right)$ negative curvature

- volume distance, d_V

The so-called volume distance is not another fundamental distance defined from the metric. It is essentially a weighted geometric combination of the line-of-sight (1/3) and transverse (2/3) comoving distances.

It was introduced in the analysis of the first detection of baryon acoustic oscillations as a way to take into account the error produced by the Alcock-Paczynski effect.

It is defined as:

$$d_V(z) = \left[d_M^2(z) \frac{cz}{H(z)}\right]^{1/3}$$

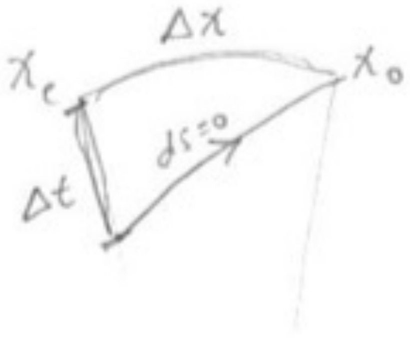
Effect of expansion

The existence of expansion (and the fact that light propagates at finite speed) implies that the emission from a source and the detection by an observer are never at the same instant of the Universe (or at the same value of the scale factor).
Hence, they are never in the same comoving frame and **the two comoving distances are not measurable distances, and may be considered "non-physical"**.

Several "physical" non-comoving distances can be defined:

- Light-travel distance, d_T

This is the segment Δt in the spacetime triangle



$$ds^2 = -dt^2 + a^2(t)\, d\chi^2$$

$$ds^2 = 0 \Rightarrow \underline{dt = a^2(t)\, d\chi^2}$$

It can be computed from,

$$\frac{\dot{a}}{a} = H_0 E(a) \quad \text{or,} \quad da = a H_0 E(a)\, dt$$

$$\text{or,} \quad t(a) = \frac{1}{H_0} \int_{a_e}^{a_0} \frac{da}{a E(a)}$$

Note that we saw that $D_c = \Delta x = \frac{1}{H_0} \int \frac{1}{a^2 E(a)}\, da$

$\Rightarrow$ there is a factor $\underline{a}$ between them, i.e.

$\underline{dt = a\, dx}$ → which is exactly what we saw above in the spacetime triangle

$D_T = c \int a(t)\, dx$ $a(t)$ varies as the light propagates

and it is a potentially measurable distance (if the time of emission is known)

- Proper distance, d_P

The proper distance is similar to the light-travel distance, but for a fixed value of the scale factor (the one of the observer), and thus it is not an observable distance, since it is not connected to an actual propagation.

It is given by $d_P = a\ d_C$,

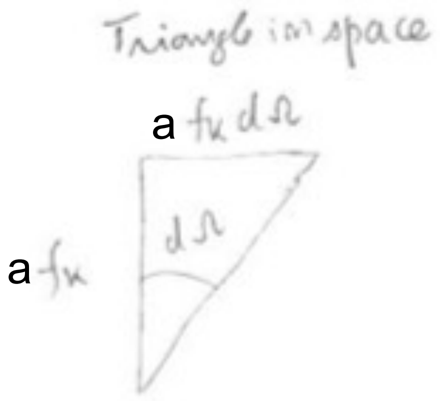
for $a = 1$, $d_P = d_C$

It is the distance that would be measured by a ruler placed between two points. Conceptually it is the simplest distance, and the one that could be thought as “the true distance”. However, it is not directly measurable.

- Angular-diameter distance, d_A

Consider a large transversal region of the Universe (meaning with all its points at the same redshift) that occupies a solid angle $d\Omega^2$ in the sky.

Like we did for the comoving diameter-angular distance, we can draw the corresponding spatial triangle for this source-observer system:



The difference with respect to that case is that due to the expansion, the intrinsic size of the observed region is not its comoving size (fixed) but its proper size (expanding).

We can thus define the (non-comoving) **angular-diameter distance d_A**

which is just: $d_A = a_{source}\, d_M$ (distance from observer at $a_0=1$ to the source at the spherical surface a_{source})

or $d_A = a_{source}\, d_M = a_{source}\, d_C$ for flat space.

We can also define the angular-diameter distance between two points (instead of between a point and the origin) :

$$D_A(z_1, z_2) = a(z_2)\, f_K\big(\chi(z_1, z_2)\big)$$

For flat → $D_A(z_1, z_2) = D_A(z_2) - D_A(z_1)$

The angular-diameter distance is the cosmological probe measured in observations of standard rulers.

A **standard ruler** is an object for which the intrinsic size is known (from theory) and for which the angular size can be observed → enabling to determine its distance d_A, which in turn contains cosmological information because it is model-dependent (and can be computed from theory).

A useful standard ruler needs to be at a:

reasonably high redshift (to contain cosmological information),

be large (for its size to expand, so cannot be an astrophysical collapsed object; and to occupy a large enough solid angle that can be measured with good precision),

be observable;

Do such objects exist in the Universe?

Yes, a **horizon** is a good candidate for this! In particular, the **sound horizon** at recombination (z=1100). The sound horizon of the Universe is observed as a peak in the CMB power spectrum (the **first peak of CMB**) and also as a peak in the matter power spectrum (the baryon acoustic oscillations **BAO peak** - even though the analysis of BAO uses d_V instead of d_A).

- Luminosity distance, d_L

As we already saw, the luminosity distance is not another fundamental distance from the metric. It is simply a version of the angular-diameter distance corrected for the effects of the redshift on the luminosity.

Consider the emission of light from a source of luminosity L, from which we measure its flux F.

$$F = \frac{L}{4\pi D_M^2} = Flux$$

note that the distance to be used here is the distance from the source to a spherical surface at a_0, i.e. the angular-diameter distance d_A with a=1, which is the comoving angular-diameter distance d_M.

Measuring the flux and knowing the intrinsic luminosity enables us to measure the comoving angular-diameter distance.

However, the luminosity in our reference-frame at a_0 is not the one emitted at the **rest-frame**. The luminosity is “redshifted” in two different ways:

$L_o = \frac{E_o}{\Delta t_o}$, but the photons are redshifted by the expansion :

$$E \propto \nu$$

$$\text{so} \quad E_0 = \frac{E_e}{1+z}$$

and the unit time dilates with the expansion :

$$\frac{\Delta t_0}{a(t_0)} = \frac{\Delta t_e}{a(t_e)}$$

$$\text{so} \quad \Delta t_0 = \Delta t_e (1+z)$$

This means that Luminosity is "redshifted" by $L_0 = \frac{L}{(1+z)^2}$ or $L = L_0 \, a^{-2}$

Luminosity dilutes faster than frequency
(a^{-2}) (a^{-1})

So in terms of rest-frame luminosity (intrinsic luminosity) we can write,

$$F = \frac{L_e}{4\pi a_0^2 D_M^2 (1+z)(1+z)}$$

The correction factors are absorbed in the definition of a **luminosity distance**, such that:

$$d_L = d_M\,(1+z) \qquad \text{or} \qquad d_L = d_C\,(1+z)\ \ \text{(flat)}$$

or, in function of the angular-diameter distance to the source:

$d_L = d_A\,(1+z)^2$ → d_L seen explicitly as a renormalization of d_A

The luminosity distance is the cosmological probe measured in observations of standard candles.

A **standard candle** is an object for which the intrinsic luminosity is known (from theory) and for which the flux can be observed → enabling us to determine its distance d_L, which in turn contains cosmological information because it is model-dependent (and can be computed from theory).

Do such objects exist in the Universe?

Yes, **supernovae of type Ia** are good candidates for this. Even though they do not all have the same intrinsic luminosity (absolute magnitude) as first thought, they can be "standardized" (meaning the observed light-curves can be shifted in a way to renormalize their absolute magnitudes).

Volume

The metric also defines volumes.

The volume, like the distances, is a geometrical quantity that can be used for cosmological tests.

The comoving volume, V_C , is the volume where the number density of objects that follow the cosmic flow remains constant as the Universe evolves.

Comoving element of Volume:

$$dV_c = dA \; dD_c$$

↘ longitudinal dimension

↓ transversal area

dA = (2D) Comoving area $= d\Omega \cdot D_M^2 = d\Omega \; D_A^2 (1+z)^2$

↳ Comoving transversal distance

and as we saw, $dD_c = \frac{c}{H_0} \frac{1}{E(z)} dz$

So the Volume element is:

(c=1)

$$\left[\, dV_C = \frac{D_A^2}{H(z)} \underline{(1+z)^2}\, d\Omega\, dz \,\right]$$

It is a ratio between D_A^2 and $H(z)$

The integrated volume, from z=0 to z and over the full angular sky is thus,

$$V_C(z) = \iiint dV_c = \frac{4}{3}\pi D_M^3 (z)$$

We can also define the proper volume, V_P , multiplying the comoving volume by a^3 :

$$dV_p = \frac{D_A^2}{H(z)} \frac{1}{1+z}\, d\Omega\, dz$$

The volume is the cosmological probe measured in number counts observations.

The **number of objects** (e.g. galaxy clusters) within a volume of the Universe (defined by an angular size and a redshift size) is counted, usually in bins of a physical property, such as mass (building a **mass function**).
If the mass function is known (from theory), the comparison between predicted and observed number counts → enables to measure the volume V_C which in turn contains cosmological information because it is model-dependent (and can be computed from theory).