The Poincaré group

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The definition of group

Definition

A group is a set of elements where you can combine any two of them and the result is still in the set — with the following four rules satisfied.

1. **Closure:** If you combine two elements, you stay inside the set.
2. **Associativity:** Order of grouping doesn't matter: $(a \star b) \star c = a \star (b \star c)$
3. **Identity:** There is a special element e that does nothing: $a \star e = a$
4. **Inverse:** Every element has an inverse: $a \star a^{-1} = e$

Example: Integers with Addition

Example: The set $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ with the operation "$+$" (usual addition).

Why is this a group?

1. **Closure:** Adding two integers gives another integer. Example: $3 + (-5) = -2$
2. **Associativity:** $(a + b) + c = a + (b + c)$ Example: $(1 + 2) + 3 = 1 + (2 + 3)$
3. **Identity:** 0 does nothing: $a + 0 = a$
4. **Inverse:** Every number has an opposite: $a + (-a) = 0$

So $(\mathbb{Z}, +)$ is a group.

Abelian Group

Abelian Group
A group G is **abelian** if its elements commute:

$$g_1 g_2 = g_2 g_1.$$

Physical meaning: Two symmetry transformations can be performed in any order without changing the result.

Examples:

- Spatial translations
- $U(1)$ phase transformations:
$$\psi \to e^{i\alpha}\psi$$

Lie Group

Lie Group

A **Lie group** is a continuous group of transformations depending smoothly on real parameters α^a:

$$U(\alpha) = e^{i\alpha^a T^a}.$$

- The T^a are generators (more later).
- Generators satisfy:

$$[T^a, T^b] = if^{abc}T^c.$$

Examples in physics:

- $SO(3)$ — rotations
- $SU(2)$ — spin
- $SU(3)$ — color symmetry

Exercises

1. Is $(\mathbb{N}, +)$ a group?
2. Nonzero real numbers $(\mathbb{R}^\times, \cdot)$ form a group?
3. Are 2×2 matrices with determinant 1 a group under multiplication? Do they form an Abelian (commutative) group?

Group representation

- A **group** describes **symmetries**.
- A **representation** tells us how those symmetries **act on vectors or states**.
- Technically:
  - each group element $\rightarrow$ a matrix (or operator);
  - group multiplication $\rightarrow$ matrix multiplication.

$$\boxed{\text{Representation is a concrete way a symmetry acts}}$$

Physical states live in vector spaces, so symmetries must be represented by operators acting on those states.

Example: Rotations of the Plane

For each angle θ, the rotation is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

It acts on vectors $v \in \mathbb{R}^2$ by matrix multiplication:

$$v \longmapsto R_\theta v$$

Why is this a group?

1. **Closure:** $R_\theta R_\varphi = R_{\theta+\varphi}$ (two rotations give another rotation)
2. **Associativity:** True because matrix multiplication is associative
3. **Identity:** $R_0 = I$ (no rotation)
4. **Inverse:** $R_\theta^{-1} = R_{-\theta}$ (rotate back)

This group is called $SO(2)$.

The Standard Representation of $SO(2)$

The group:

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

Representation:

$$\rho : SO(2) \longrightarrow GL(2, \mathbb{R})$$

$$\rho(A) = A$$

This is called the **standard representation**: the group acts on $\mathbb{R}^2$ by rotating vectors.

Is $GL(2,\mathbb{R})$ a Representation of $U(1)$?

The unitary group $U(n)$ is a real Lie group of dimension n^2. In the simple case $n = 1$, the group $U(1)$ corresponds to the circle group, isomorphic (isomorphic groups are algebraically identical, sharing the same structure and properties despite having different elements or operation definitions) to the set of all complex numbers that have absolute value 1, under multiplication.

$GL(2,\mathbb{R})$ is the General Linear Group of degree 2 over the real numbers, defined as the set of all matrices with real entries that have a non-zero determinant.

1. What a representation is
A representation of $U(1)$ on $\mathbb{R}^2$ is a homomorphism (mapping preserves the structure)

$$\rho : U(1) \to GL(2,\mathbb{R}), \qquad \rho(g_1 g_2) = \rho(g_1)\rho(g_2).$$

It is *not* the whole group $GL(2,\mathbb{R})$, but only the image $\rho(U(1))$.

2. Example: 2D real representation

$$\rho(e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

These matrices form a subgroup isomorphic to $SO(2) \cong U(1)$.

3. Why this works

All matrices are generated by a single matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(e^{i\theta}) = e^{\theta J}. \quad \textit{show}!$$

Since everything is built from the same generator, all matrices commute.

4. Complex vs real representations

Over $\mathbb{C}$, the same matrices diagonalize:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \chi_{+1} \oplus \chi_{-1}.$$

So over $\mathbb{C}$ the representation is reducible (more later).

5. Conclusions

- $GL(2,\mathbb{R})$ itself is not a representation.
- A representation is a homomorphism into it.
- The image of $U(1)$ is always abelian.
- Over $\mathbb{C}$, all irreducible representations of $U(1)$ are 1D.

Exercises

1. Show that a 2D rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

satisfies $R^T R = I$ and $\det R = 1$.

2. How many parameters does $SO(3)$ have?

3. Show that matrices U satisfying

$$U^\dagger U = I, \quad \det U = 1$$

form a group.

Exercises

1. Expand $U = e^{i\alpha T}$ to first order in α^a.
2. Show that if T is Hermitian, then $U = e^{i\alpha T}$ is unitary.
3. The Pauli matrices satisfy:

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k.$$

   Show that $T^i = \sigma^i/2$ satisfy

$$[T^i, T^j] = i\epsilon^{ijk} T^k.$$

4. How many generators does $SU(2)$ have?

Exercises

1. Show that the identity element must be represented by the identity matrix.
2. Show that $\rho(g^{-1}) = \rho(g)^{-1}$.
3. For $SU(2)$ with generators T^a, write the infinitesimal representation:

$$U(\alpha) \approx 1 + i\alpha^a T^a.$$

Solutions: Identity Element Represented by the Identity Matrix

Let

$$\rho : G \to GL(V)$$

be a group representation. Then

$$\rho(e) = 1,$$

where e is the identity element of G.

Proof
Since ρ is a homomorphism:

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2).$$

Take $g_1 = e$ and $g_2 = g$:

$$\rho(eg) = \rho(e)\rho(g).$$

But $eg = g$, so:

$$\rho(g) = \rho(e)\rho(g).$$

Multiply on the right by $\rho(g)^{-1}$:

$$\rho(e) = 1.$$

Conclusion
The identity of the group must be represented by the identity operator in any representation.

Solutions: Infinitesimal Representation of $SU(2)$

Generators of $SU(2)$
Let T^a $(a = 1, 2, 3)$ be the generators of $SU(2)$ in some representation, satisfying the algebra:

$$[T^a, T^b] = i\epsilon^{abc} T^c.$$

Infinitesimal group element
For a small rotation parametrized by α_a:

$$U(\vec{\alpha}) \in SU(2), \quad |\vec{\alpha}| \ll 1,$$

the exponential map gives:

$$U(\vec{\alpha}) = e^{i\alpha_a T^a}.$$

First-order (infinitesimal) expansion

$$\boxed{U(\vec{\alpha}) \approx 1 + i\alpha_a T^a + \mathcal{O}(\alpha^2)}$$

This is the standard infinitesimal representation of $SU(2)$.

Remarks

- The generators T^a determine how states transform under infinitesimal rotations.
- Finite rotations are recovered via the full exponential $U = e^{i\alpha_a T^a}$.

Irreducible representations

Irreducible representations ("irreps") in group theory are the fundamental, non-decomposable building blocks of group representations that cannot be broken down into smaller, independent sub-representations. In very simple terms:

- A representation can sometimes be **split** into smaller ones.
- If it *cannot* be split, it is called irreducible.
- Physically:
  - irreducible representations describe **elementary systems**;
  - reducible ones describe **composite systems**.

Reducible vs. irreducible: rotations in the plane

- Consider the group of rotations in the plane, $SO(2)$.
- **Irreducible example:**
  - Action on the 2D plane (x, y);
  - Rotation by an angle θ:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

  - This representation cannot be decomposed into smaller invariant subspaces.
- **Reducible example:**
  - Action on two independent planes (x_1, y_1) and (x_2, y_2);
  - The rotation acts block-diagonally:

$$R(\theta) \oplus R(\theta).$$

  - This representation can be split into two copies of the same one.

Generators of a group: a simple example

- Consider the group $(A, *_9)$, where
$$A = \{1, 2, 4, 5, 7, 8\},$$
and $*_9$ means multiplication modulo 9.
- This is a group because:
  - it contains the identity: 1;
  - every element has an inverse modulo 9; what is the inverse of 4?
  - multiplication modulo 9 is associative.
- A **generator** is an element from which all others can be obtained.
- Example:
$$2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 7, \quad 2^5 = 5, \quad 2^6 = 1 \pmod 9$$
- Therefore,
$$\boxed{\langle 2 \rangle = A}$$
and the group is cyclic - all elements can be obtained from one. Will 5 do the job?

What are Casimir invariants?

- Symmetries in physics are described by **groups**.
- Each group has **generators** (think of them as symmetry operations).
- A Casimir invariant is a special combination of generators that:
  - commutes with *all* generators;
  - has the *same value* for all states in a given representation.

Casimir invariant is a quantity that labels a representation

Physical meaning

Casimir invariants correspond to **observable quantities** such as mass or spin.

Casimir invariants: a concrete example

- Consider the **rotation group** $SO(3)$.
- Its generators are the components of **angular momentum**:

$$\vec{J} = (J_x, J_y, J_z).$$

- Individual components do *not* commute:

$$[J_x, J_y] \neq 0.$$

- But the combination

$$\boxed{J^2 = J_x^2 + J_y^2 + J_z^2}$$

commutes with all generators.

J^2 is a Casimir invariant of the rotation group.

Angular Momentum as Generators

The components of angular momentum J_i are the generators of $SO(3)$ because they are the infinitesimal operators corresponding to rotations around the x, y, z axes.

Through the exponential map,

$$U(R) = e^{-\,i\theta^i J_i},$$

these operators generate finite 3D rotations.
They form the basis of the Lie algebra $\mathfrak{so}(3)$, defining how quantum states change under rotations.

Why Angular Momentum Generates Infinitesimal Rotations

Angular momentum operators represent the *rate of change* of a system under an infinitesimal rotation.

Commutation Relations

The components satisfy:

$$[J_i, J_j] = i\epsilon_{ijk} J_k,$$

which is isomorphic to the commutation relations of $SO(3)$.

Symmetry Principle

Conservation of angular momentum follows from the invariance of physical laws under spatial rotations.

Exercises: Angular Momentum and Rotations

1. Infinitesimal Rotations
Consider a small rotation by an angle $\delta\theta$ around the z-axis:

$$U(\delta\theta) = 1 - i\delta\theta\, J_z.$$

(a) Apply this operator to a state $|\psi\rangle$ and compute the change $\delta|\psi\rangle$.

(b) Interpret J_z as the generator of rotations.

2. Commutation Relations
Given

$$[J_i, J_j] = i\epsilon_{ijk} J_k,$$

(a) Show that the algebra closes.

3. Finite Rotations
Show that a finite rotation around a unit vector $\hat{n}$ is:

$$U(R) = e^{-i\theta \hat{n}\cdot\vec{J}}.$$

Expand to second order in θ and verify consistency with two successive infinitesimal rotations.

4. Casimir Operator
Define:

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2.$$

Show that:

$$[\vec{J}^2, J_i] = 0.$$

What does this imply about simultaneous eigenstates?

1(a) Action on a state
For a small rotation around z:

$$U(\delta\theta) = 1 - i\delta\theta\, J_z.$$

Acting on $|\psi\rangle$:

$$|\psi'\rangle = (1 - i\delta\theta J_z)|\psi\rangle.$$

Change in the state:

$$\delta|\psi\rangle = |\psi'\rangle - |\psi\rangle = -i\delta\theta J_z|\psi\rangle.$$

Interpretation
J_z gives the *rate of change* of the state under an infinitesimal rotation.
Hence J_z is the generator of rotations about z.

Solutions: Commutation Relations

Given:

$$[J_i, J_j] = i\epsilon_{ijk} J_k.$$

1. Closure

The commutator of two generators is again a linear combination of generators. Therefore the algebra closes.

2. Relation to $SO(3)$

The Lie algebra of $SO(3)$ satisfies:

$$[L_i, L_j] = i\epsilon_{ijk} L_k.$$

Thus the angular momentum algebra is isomorphic to $\mathfrak{so}(3)$.

Solutions: Finite Rotations

We consider spatial rotations generated by the angular momentum operators J^i, satisfying the algebra

$$[J^i, J^j] = i\epsilon^{ijk} J^k .$$

(a) Finite rotation operator
A finite rotation by an angle θ about a unit vector $\hat{n}$ is defined as

$$U(R) = \exp(-i\theta\, \hat{n} \cdot \mathrm{J})\,,$$

where

$$\hat{n} \cdot \mathrm{J} = n_i J^i .$$

This follows from the general principle that continuous symmetries are generated by exponentiating their infinitesimal generators. Indeed, for an infinitesimal rotation $\delta\theta$,

$$U(\delta\theta) = 1 - i\delta\theta\, \hat{n} \cdot \mathrm{J} + \mathcal{O}(\delta\theta^2).$$

Successive application of N such infinitesimal rotations with $\delta\theta = \theta/N$ gives

$$U(R) = \lim_{N\to\infty} \left(1 - i\frac{\theta}{N}\hat{n} \cdot \mathrm{J}\right)^N = \exp(-i\theta\hat{n} \cdot \mathrm{J})\,.$$

(b) Expansion to second order
Expand the exponential:

$$U(R) = 1 - i\theta\hat{n} \cdot \mathrm{J} - \frac{\theta^2}{2}(\hat{n} \cdot \mathrm{J})^2 + \mathcal{O}(\theta^3).$$

Thus,

$$U(R) = 1 - i\theta n_i J^i - \frac{\theta^2}{2} n_i n_j J^i J^j + \mathcal{O}(\theta^3).$$

(c) Two successive infinitesimal rotations
Now consider two successive infinitesimal rotations of size $\delta\theta$ about the same axis:

$$U_1 = 1 - i\delta\theta\hat{n}\cdot \mathrm{J}, \quad U_2 = 1 - i\delta\theta\hat{n}\cdot \mathrm{J}.$$

Their product is:

$$U_2U_1 = (1 - i\delta\theta\hat{n}\cdot \mathrm{J})\,(1 - i\delta\theta\hat{n}\cdot \mathrm{J})\,.$$

Multiplying:

$$= 1 - 2i\delta\theta\hat{n}\cdot \mathrm{J} - (\delta\theta)^2(\hat{n}\cdot \mathrm{J})^2.$$

Now set $\theta = 2\delta\theta$. Then:

$$U_2U_1 = 1 - i\theta\hat{n}\cdot \mathrm{J} - \frac{\theta^2}{2}(\hat{n}\cdot \mathrm{J})^2 + \mathcal{O}(\theta^3).$$

This agrees exactly with the second-order expansion of the exponential:

$$\exp(-i\theta\hat{n}\cdot \mathrm{J}).$$

Conclusion
The finite rotation operator is

$$\boxed{U(R) = e^{-i\theta\hat{n}\cdot \mathrm{J}}}$$

and its second-order expansion is consistent with the composition of successive infinitesimal rotations, confirming that the generators J^i correctly generate finite rotations via exponentiation.

Solutions: Casimir Operator

Define:

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2.$$

Compute:

$$[\vec{J}^2, J_i] = [J_j J_j, J_i].$$

Using:

$$[J_j, J_i] = i\epsilon_{jik} J_k,$$

one finds (after applying the product rule):

$$[\vec{J}^2, J_i] = 0.$$

Implication
$\vec{J}^2$ and one component (usually J_z) can be diagonalized simultaneously.
States are labeled by:

$$|j, m\rangle, \quad \vec{J}^2 |j, m\rangle = j(j+1)|j, m\rangle, \quad J_z |j, m\rangle = m|j, m\rangle.$$

The Poincaré Group

Goal: Describe all coordinate transformations that preserve the Minkowski spacetime interval

$$ds^2 = \eta_{\mu\nu} x^\mu x^\nu, \quad \eta = \mathrm{diag}(-1, 1, 1, 1).$$

General transformation:

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu$$

- $\Lambda^\mu{}_\nu$ = Lorentz transformation
- a^μ = spacetime translation

Lorentz condition (interval invariance):

$$\Lambda^T \eta \Lambda = \eta$$

Interpretation:

- Rotations and boosts mix space and time
- Translations shift the origin of spacetime

Examples and Structure

1. Rotation (around z-axis)

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

2. Boost (along x with velocity v)

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$
$$x' = \gamma(x - vt),$$

Group structure:

$$\text{Poincaré} = \text{Lorentz} \ltimes \text{Translations}$$

10 generators:

$$\underbrace{3}_{\text{rotations}} + \underbrace{3}_{\text{boosts}} + \underbrace{4}_{\text{translations}}$$

Generators and Conserved Quantities

Continuous symmetries $\Rightarrow$ **conserved quantities** (Noether's theorem)

Spacetime translations

$\Rightarrow$ Energy and Momentum conservation

Spatial rotations

$\Rightarrow$ Angular momentum conservation

Boosts

$\Rightarrow$ Conservation of center-of-mass motion

Why the Poincaré group matters

Fundamental symmetry group of relativistic physics.

Any relativistic quantum theory must:

- admit a representation of the Poincaré group;
- classify states according to that representation.

- Continuous symmetries $\Rightarrow$ generators.
- The Poincaré group has 10 generators:
  - **4 translations**: P^μ
    - energy and momentum;
  - **6 Lorentz generators**: $M^{\mu\nu}$
    - 3 rotations;
    - 3 boosts.

Lorentz Algebra: Rotations and Boosts

Explicit decomposition for the Lorentz group

$$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk} \qquad K^i = M^{0i}$$

- J^i generate spatial rotations
- K^i generate boosts

Commutation relations

1. **Rotation–rotation:** $[J_i, J_j] = i\epsilon_{ijk}J_k$
2. **Rotation–boost:** $[J_i, K_j] = i\epsilon_{ijk}K_k$
3. **Boost–boost:** $[K_i, K_j] = -i\epsilon_{ijk}J_k$

The minus sign in $[K_i, K_j]$ distinguishes $\mathfrak{so}(1,3)$ from $\mathfrak{so}(4)$.

Why Two Boosts Produce a Rotation (Thomas Precession)

Key algebraic clue

$$[K_i, K_j] = -i\epsilon_{ijk} J_k .$$

Two boosts in different directions do not commute. Their commutator produces a rotation.

Geometric picture

- A boost changes both time and space directions.
- Performing a boost along $\hat{x}$, then along $\hat{y}$, does not give the same result as reversing the order.
- The mismatch is precisely a spatial rotation.

Infinitesimal example
Consider small boosts with parameters $\delta\eta_x$ and $\delta\eta_y$:

$$(1 - i\delta\eta_x K_x)(1 - i\delta\eta_y K_y) - (x \leftrightarrow y).$$

Using the commutator:

$$\sim -\delta\eta_x\delta\eta_y[K_x, K_y] = i\delta\eta_x\delta\eta_y J_z .$$

Result: a rotation about $\hat{z}$.

Physical consequence
For an accelerating particle (e.g. electron in an atom), successive boosts along changing directions produce a small rotation of the spin axis: **Thomas precession** (Comparison with Foucault's pendulum Thomas Precession).

Infinitesimal Lorentz Transformations

Step 1: Expand around the identity
For transformations close to the identity:

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu} \qquad |\omega| \ll 1$$

Step 2: Impose Lorentz condition

$$\Lambda^T \eta \Lambda = \eta$$

Keeping only terms linear in ω:

$$\eta_{\rho\sigma}(\delta^{\rho}_{\ \mu} + \omega^{\rho}_{\ \mu})(\delta^{\sigma}_{\ \nu} + \omega^{\sigma}_{\ \nu}) = \eta_{\mu\nu}$$

$$\Rightarrow \eta_{\mu\nu} + \eta_{\rho\nu}\omega^{\rho}_{\ \mu} + \eta_{\mu\sigma}\omega^{\sigma}_{\ \nu} = \eta_{\mu\nu}$$

Cancel zeroth order terms:

$$\eta_{\rho\nu}\omega^{\rho}_{\ \mu} + \eta_{\mu\sigma}\omega^{\sigma}_{\ \nu} = 0$$

Lowering an index:

$$\boxed{\omega_{\mu\nu} = -\omega_{\nu\mu}}$$

Conclusion: the parameters are antisymmetric.

$\Rightarrow$ 6 independent parameters

How the Lorentz Generators Arise

1. Infinitesimal Lorentz transformation

$$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu}, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

A scalar field transforms as

$$\phi'(x) = \phi(\Lambda^{-1} x)$$

For infinitesimal transformations:

$$\phi'(x) = \phi(x) - \omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} \phi(x)$$

This can be written as

$$\phi'(x) = \left(1 - \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}\right) \phi(x)$$

where the generators acting on scalar fields are

$$\boxed{M^{\mu\nu} = i(x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu})}$$

Expanding to first order:

$$\delta\phi(x) = -\frac{i}{2}\, \omega_{\mu\nu}\, (x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu}) \phi(x).$$

2. Orbital generators
We identify the differential operator:

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu).$$

These are the generators in the scalar (orbital) representation.
3. General fields
For fields with spin:

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}.$$

- $L^{\mu\nu}$: orbital part (acts on spacetime dependence)
- $S^{\mu\nu}$: spin part (acts on internal indices)

Example (Dirac field):

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

Interpretation
Generators arise as the coefficients of the infinitesimal parameters $\omega_{\mu\nu}$ in the first-order expansion of the symmetry transformation.

Casimir invariants of the Poincaré group

- In relativistic physics, symmetries are described by the **Poincaré group**.
- Casimir invariants commute with *all* generators.
- The Poincaré group has two independent Casimirs:
  - First Casimir:
  $$P^\mu P_\mu = m^2$$
  - Second Casimir (Pauli–Lubanski vector):
  $$W^\mu W_\mu = -m^2 s(s+1)$$

Casimir invariants are mass and spin

The spin s labels how states transform under rotations in the particle's rest frame.

Spin as a representation of the rotation group

- Physical rotations in space form the group $SO(3)$.
- In quantum mechanics, states transform under $SU(2)$, not directly under $SO(3)$.
- $SU(2)$ is the **double cover** of $SO(3)$:
  - each rotation in space corresponds to two elements of $SU(2)$;
  - a 360° rotation gives a minus sign for half-integer spin.
- **Spin** labels the irreducible representations of $SU(2)$:

$$s = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \ldots$$

- Each spin-s representation has dimension:

$$\boxed{2s + 1}$$

Spin tells us *how quantum states rotate*.

$SU(2)$ as the Double Cover of $SO(3)$

To every $R \in SO(3)$ correspond two elements $\pm U \in SU(2)$.

Construction via Pauli matrices
For a rotation by angle θ around unit vector $\hat{n}$:

$$U(\theta, \hat{n}) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} \in SU(2).$$

It induces a rotation of vectors through:

$$U\,(\vec{x}\cdot\vec{\sigma})\,U^{\dagger} = (R\vec{x})\cdot\vec{\sigma}.$$

Thus each U defines an $R \in SO(3)$.

Example: 2π rotation
Take $\theta = 2\pi$:

$$U(2\pi, \hat{n}) = e^{-i\pi\hat{n}\cdot\vec{\sigma}} = -I.$$

But in $SO(3)$:

$$R(2\pi) = I.$$

Hence:

$$U = +I \quad \text{and} \quad U = -I$$

both correspond to the same rotation in $SO(3)$.

Conclusion
The map $SU(2) \to SO(3)$ is two-to-one:

Physical Consequence: Spin-$\frac{1}{2}$ and 4π Rotations

Spinors transform under $SU(2)$
For a rotation by angle θ around $\hat{n}$:

$$U(\theta, \hat{n}) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}.$$

2π **rotation**

$$U(2\pi, \hat{n}) = e^{-i\pi\hat{n}\cdot\vec{\sigma}} = -I.$$

Thus for a spinor state $|\psi\rangle$:

$$|\psi\rangle \longrightarrow -|\psi\rangle.$$

A 2π rotation changes the *sign* of the state.

4π **rotation**

$$U(4\pi, \hat{n}) = e^{-i2\pi\hat{n}\cdot\vec{\sigma}} = +I.$$

Only after a 4π rotation does the state return to itself.

Physical meaning

- Spin-$\frac{1}{2}$ particles live in representations of $SU(2)$.
- Classical vectors transform under $SO(3)$.
- The double cover explains why fermions require 4π to return to the same quantum state.

Wigner's Classification of Elementary Particles

Spacetime symmetry
The symmetry group of relativistic quantum theory is the **Poincaré group**:

$$\text{Lorentz transformations } + \text{ translations.}$$

Generators:

$$P^\mu \quad \text{(translations)}, \qquad M^{\mu\nu} \quad \text{(Lorentz)}.$$

Elementary particles correspond to **unitary irreducible representations (irreps)** of the Poincaré group.

Casimir operators (classification labels)
Two independent invariants:

$$P^\mu P_\mu = m^2 \qquad\qquad W^\mu W_\mu \quad \text{(Pauli–Lubanski vector)}$$

where

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}.$$

Massive particles ($m > 0$)
Choose rest frame $P^\mu = (m, 0, 0, 0)$.
Little group:

$$SO(3) \sim SU(2).$$

Representations labeled by:

$$m, \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

Spin s determines dimension $2s + 1$.

Massless particles ($m = 0$)
No rest frame.
Little group:

$$ISO(2).$$

Physical irreps labeled by:

$$m = 0, \qquad \text{helicity } h.$$

Helicity = projection of angular momentum along momentum:

$$h = \frac{\vec{J} \cdot \vec{P}}{|\vec{P}|}.$$

Conclusion
An elementary particle is nothing but a unitary irrep of the Poincaré group, labeled by:

$$\boxed{(m, s)} \quad \text{or} \quad \boxed{(0, h)}.$$

Concrete examples: spin-0, spin-$\frac{1}{2}$ and spin-1

- Spin labels irreducible representations of $SU(2)$.
- **Spin-0 (scalar):**
  - Dimension $2s + 1 = 1$;
  - State does not change under rotations.
- **Spin-$\frac{1}{2}$ (spinor):**
  - Dimension $2s + 1 = 2$;
  - State picks up a minus sign under 360° rotation.
- **Spin-1 (vector):**
  - Dimension $2s + 1 = 3$;
  - Transforms like an ordinary vector under rotations.

From spin to fields

- In quantum field theory, particles are described by **fields**.
- The spin of a particle determines the type of field used to describe it.
- **Spin-0 $\rightarrow$ Scalar fields**
  - one component;
  - example: Higgs field.
- **Spin-$\frac{1}{2}$ $\rightarrow$ Spinor fields**
  - two (Weyl) or four (Dirac) components;
  - example: electrons, quarks.
- **Spin-1 $\rightarrow$ Vector fields**
  - three physical polarizations (for massive particles);
  - example: photon, W, Z bosons.

Spin fixes how fields transform under Lorentz transformations.

How Fields Transform

A Poincaré transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

Field transformation rule:

$$\phi'(x) = \rho(\Lambda)\,\phi\big(\Lambda^{-1}(x-a)\big)$$

- Translation $\Rightarrow$ shift of argument
- Lorentz transformation $\Rightarrow$ internal mixing
- $\rho(\Lambda)$ = representation of Lorentz group

Examples of Representations

Scalar field $\quad \phi'(x) = \phi(\Lambda^{-1}(x-a))$

Vector field $\quad A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}(x-a))$

Spinor field $\quad \psi'(x) = S(\Lambda)\psi(\Lambda^{-1}(x-a))$

Generators: Translations

Infinitesimal translation:

$$x^\mu \to x^\mu + \epsilon^\mu$$

Field variation:

$$\delta\phi(x) = -\epsilon^\mu \partial_\mu \phi(x)$$

Translation generator

$$P^\mu = i\partial^\mu$$

Interpretation:

- Time translations $\Rightarrow$ Energy
- Space translations $\Rightarrow$ Momentum

Lorentz Generators: Structure and Origin

Infinitesimal Lorentz transformation:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Coordinates transform as:

$$x^{\mu} \to x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu}$$

Field transformation:

$$\phi'(x) = \phi(x) + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} \phi(x)$$

$$\delta\phi(x) = \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} \phi(x)$$

Orbital and Spin Contributions

Two effects under Lorentz transformations:

1. **Coordinate change** $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$
2. **Internal index mixing** (depends on representation)

Total generator

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$$

Orbital part (from coordinate shift)

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

Acts on *all* fields.

Spin part (from representation)

$$S^{\mu\nu} = \begin{cases} 0 & \text{scalar} \\ \frac{i}{4}[\gamma^\mu, \gamma^\nu] & \text{Dirac spinor} \\ i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma) & \text{vector} \end{cases}$$