Cosmologia Física

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Structure formation

Relativistic perturbed equations

General Relativistic Treatment

In General Relativity, we need to consider perturbations in the cosmological fluid and in the metric.

We want to find a set of equations to study the time-evolution of any perturbed quantity (for each scale), and in particular we are interested in the time-evolution of the density contrast of the matter field (the basis for structure formation).

The perturbed quantities may be written as independent modes in harmonic space and the modes evolve independently in the linear regime → no spatial evolution in the linearized equations.

The set of equations are the (perturbed and linearized) Einstein equations plus energy-momentum conservation equations.

The energy-momentum conservation equation is a continuity equation in the case of a perfect fluid. In the more complex case of a system of particles with an energy distribution evolving in the phase space, the conservation equation is the (perturbed) Boltzmann equation.

The set of equations are called the Einstein-Boltzmann equations.

Metric Perturbations

It is convenient to write the diagonal Robertson-Walker (RW) metric using conformal time τ, i.e., to factorize the scale factor:

$$ds^2 = a^2(\tau)\left[-d\tau^2 + dr^2 + f_K^2(r)\, d\Omega^2\right] \qquad dt = a\, d\tau \qquad \frac{d}{dt} = \frac{1}{a}\frac{d}{d\tau}$$

we also define the **conformal Hubble function** $\mathcal{H}$ = **a H** = a' / a that is useful when considering derivatives with respect to conformal time.

In matrix form and using cartesian coordinates (and in the case of flat space), the metric is thus,

$$\bar{g}_{\mu\nu} \equiv a^2 \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \delta_{ij} \end{pmatrix}$$

where δ_{ij} is the Kronecker delta, i.e., the identity matrix, and i,j (spatial indexes) run from 1 to 3.

Now, the inhomogeneities in the density field (and in other sources of gravity) produce a change in the metric.

The metric becomes inhomogeneous and, if the modifications are small, it is usually written as a perturbation to Robertson-Walker metric →

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

RW is called the background metric in the inhomogeneous universe.

In general, we can perturb all 10 components of the symmetric 4x4 RW metric, and we may write a general (symmetric) 4x4 metric perturbation as:

$$\delta g_{\mu\nu} \equiv a^2 \begin{pmatrix} \overset{\text{S}}{-2\phi} & \overset{\text{V}}{w_i} \\ \underset{\text{V}}{w_i} & \underset{\text{T}}{-2\Psi\delta_{ij} + 2h_{ij}} \end{pmatrix}$$

This introduces 10 new random fields (which in principle, if they are independent, are expected to introduce 10 new degrees of freedom in the metric): 1S + 3V + 6T

These 10 new quantities are:

Φ (1 scalar in the component tt) ;
w_i (a vector with 3 components ti) ;
h_{ij} (a traceless tensor with 5 components ij);
ψ (the trace of the spatial tensor of the perturbed metric)

Here, the tensor was written as a traceless tensor + trace, i.e, the types of the 10 components are now: 2S + 3V + 5T

The vector and tensor components can be further decomposed as we will see next: the SVT decomposition.

This is useful because S, V and T perturbations will evolve independently of each other, and so it will simplify the system of differential Einstein equations.

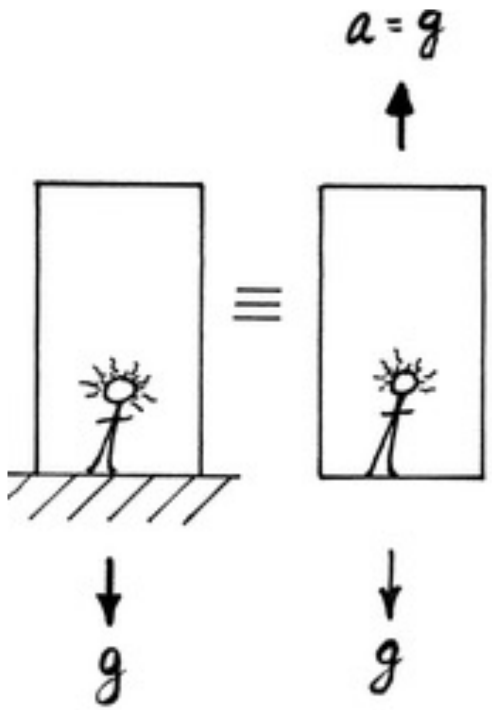
Scalar perturbations

What is the meaning of the two metric scalar perturbations?

Φ - From the **equivalence principle :**

Gravitational field (gravitational mass) ←→ Acceleration of the reference system (inertial mass)

In the well-known “gedanken” experiment in special relativity:



Photon travel time from ceiling to floor t = h/c
Floor’s velocity increased by g h/c
Frequency shift $\Delta\nu/\nu = \Delta v/c = gh/c^2$
Time dilation $\Delta t/t = gh/c^2$
Equivalence principle → time dilation = $\Delta\Phi / c^2$

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + dx^2$$

Minkowski metric in an accelerated frame (or with a gravitational potential: a metric perturbation)

ψ - In GR, **spatial curvature** also contributes to gravity → a perturbation to spatial curvature also changes the dynamics (**length contraction**)

$$ds^2 = - c^2 dt^2 + \left(1 + \frac{2\Psi}{c^2}\right) dx^2$$

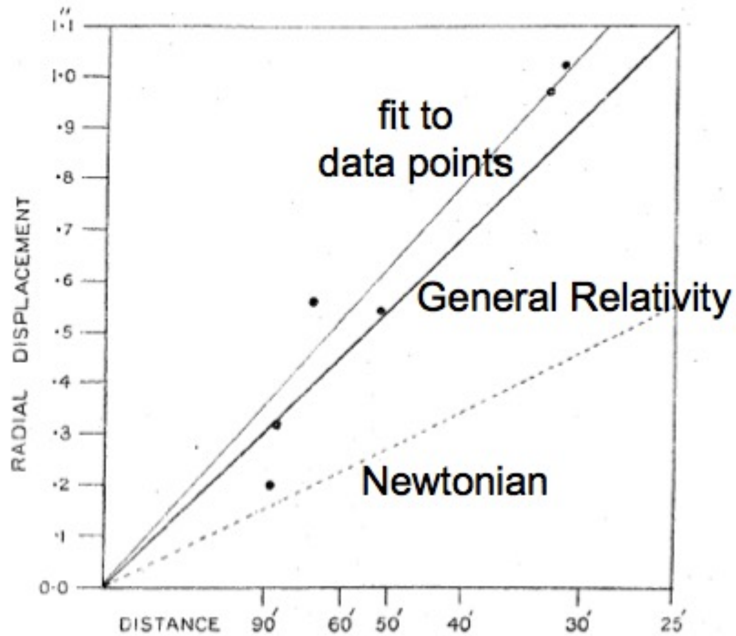
The simplest inhomogeneous metric is the scalar one - it includes two potentials, defining a **space-time curvature** that describes gravitational effects at first-order:

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 + \frac{2\Psi}{c^2}\right) dx^2$$

Null geodesics are determined by **Φ + ψ** (= 2Φ if they are equal),

while in Newtonian gravity, trajectories are determined by Φ (potential associated to the inertial mass), and curvature is not considered

→ **light deflection in GR is twice as large than in Newtonian gravity** → first test of GR (1919 eclipse)



Vector perturbations

Any vector can be decomposed in a sum of 2 special types of vectors:

a gradient vector
and a divergence-free vector:

$$w_i = w_{;i} + w_i^{\perp}$$

Remember ';i' stands for derivative in curved space-times , i.e., the covariant derivative with respect to the coordinate 'i'.
This is computed as the partial derivative ',i' with respect to 'i' plus the appropriate contractions with the connection.
For example, the covariant derivative of a vector is: $v^a;b = v^a,b + \Gamma^a_{cb} v^c$

The decomposition leads to two important features:

i) The 3 components of the gradient vector $w_{;i}$ are all computed from derivatives of the same scalar w (this is usually called the potential of the associated vector field)
→ **the three components contain only 1 independent quantity, a scalar w**

ii) The 3 components of the divergence-free vector $w_i^{\perp}$ are also not all independent, they are related due to the divergence-free nature of the vector, i.e.,

$$w^{\perp 1}{}_{;1} + w^{\perp 2}{}_{;2} + w^{\perp 3}{}_{;3} = 0$$

→ the 3D divergence-free vector **has only 2 independent vector components**

So, the vector perturbation with 3 components 3V was decomposed in 1S+2V

Tensor perturbations

Any traceless tensor can be decomposed in a sum of 3 special types of tensors:

a gradient of a gradient, i.e., a laplacian tensor,
a gradient of a divergence-free vector,
and a divergence-free and traceless tensor:

$$h_{ij} = D_{ij} h + h_{(i;j)} + h^T_{ij} \qquad \text{with} \quad D_{ij} \equiv \nabla_i \nabla_j - \frac{1}{3}\delta_{ij}\Delta$$

Let us look at the 3 terms:

i) In the **first term**, the tensor is $\nabla_i \nabla_j h = \begin{bmatrix} h_{;11} & h_{;12} & h_{;13} \\ h_{;21} & h_{;22} & h_{;23} \\ h_{;31} & h_{;32} & h_{;33} \end{bmatrix}$

and then the diagonal terms are subtracted by $\frac{1}{3}\left(h_{;11} + h_{;22} + h_{;33}\right)$

This results in a tensor D_{ij} h that is traceless.

Note that the full 3x3 symmetric tensor defined by the first term (containing in principle 6 independent quantities) is all built from 1 single quantity, the scalar h → **it contains only 1 independent quantity, the scalar h**

ii) The **second term** is a tensor built from a vector h_i :

$$h_{(i;j)} = \frac{1}{2}\begin{bmatrix} (h_{1;1} + h_{1;1}) & (h_{1;2} + h_{2;1}) & (h_{1;3} + h_{3;1}) \\ (h_{2;1} + h_{1;2}) & (h_{2;2} + h_{2;2}) & (h_{2;3} + h_{3;2}) \\ (h_{3;1} + h_{1;3}) & (h_{3;2} + h_{2;3}) & (h_{3;3} + h_{3;3}) \end{bmatrix}$$

so, in principle it has 3 independent vector quantities.

However, we will consider that the vector h_i is divergence-free, i.e.

$$h_{1;1} + h_{2;2} + h_{3;3} = 0$$

This makes $h_{(i;j)}$ traceless → **the second term only contains 2 independent quantities**

iii) The **third term** is a traceless and divergence-free tensor h^T_{ij} .

This means that its 6 independent components are constrained by 4 equations:

$$h^T_{11} + h^T_{22} + h^T_{33} = 0$$

$$h^T_{11;1} + h^T_{12;2} + h^T_{13;3} = 0$$
$$h^T_{21;1} + h^T_{22;2} + h^T_{23;3} = 0$$
$$h^T_{31;1} + h^T_{32;2} + h^T_{33;3} = 0$$

→ it contains only 2 independent quantities

So, the (traceless) tensor perturbation with 5 components 5T was decomposed in 1S+2V+2T

This tensor decomposition was made for a traceless tensor.

In general, the tensor perturbation do not need to be traceless. Now that the decomposition is made, it is very easy to generalize it to the case of a non-zero trace.
We just to need to add a **trace tensor**, i.e., a diagonal tensor only with the trace information → the trace is then an additional degree of freedom, and it is a scalar.

The trace tensor is usually written as $-2\,\psi\,\delta_{ij}$ → **it contains only 1 independent quantity, the trace (the scalar ψ)**

So, the tensor perturbation with 6 components 6T was decomposed in 2S+2V+2T

Collecting all terms, the metric SVT perturbations are:

$$\begin{aligned}\delta g_{\mu\nu} &= \delta g^S_{\mu\nu} + \delta g^V_{\mu\nu} + \delta g^T_{\mu\nu} \\ &= a^2 \begin{pmatrix} -2\phi & w_{;i} \\ w_{;i} & -2\psi\delta_{ij} + 2h_{;ij} \end{pmatrix} + a^2 \begin{pmatrix} 0 & w_i^\perp \\ w_i^\perp & 2h_{(i;j)} \end{pmatrix} + a^2 \begin{pmatrix} 0 & 0 \\ 0 & 2h^T_{ij} \end{pmatrix}\end{aligned}$$

type	fields	constraints	degrees of freedom
scalar perturbations	ϕ, ψ, w, h	-	4
vector perturbations	$w_i^\perp,\ h_i$	$\nabla^i w_i^\perp = \nabla^i h_i = 0$	4
tensor perturbations	h^T_{ij}	$\nabla^i h^T_{ij} = (h^T)^i_i = 0,\ h^T_{ij} = h^T_{ji}$	2

So, the fundamental types of the 10 degrees of freedom are:

4S + 4V + 2T

instead of 1S + 3V + 6T

Energy-Momentum Tensor Perturbations

The homogeneous metric is sourced by a perfect fluid:

$$T_{\mu\nu} = (\rho + p)\, u_\mu\, u_\nu + p\, g_{\mu\nu}$$

where u is the fluid 4-velocity $\quad u^\mu u_\nu = -1$

A perfect fluid has no heat conduction q (a 0i vector) and no anisotropic stress π (a ij tensor). A more general fluid is:

$$T_{\mu\nu} = (\rho + p)\, u_\mu\, u_\nu + p\, g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu}$$

The perturbed metric is sourced by a perturbed fluid:

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu = [\bar{\rho}(1+\delta) + \bar{p}(1+\delta_p)]\, u^\mu\, u_\nu + [\bar{p}(1+\delta_p)]\, g^\mu_\nu$$
$$+ [(\bar{\rho}+\bar{p})\delta u_\nu + q_\nu]\; u^\mu + [(\bar{\rho}+\bar{p})\delta u^\mu + q^\mu]\; u_\nu + \pi^\mu_\nu$$

The source quantities include components of a perturbed perfect fluid:

-- δ - density perturbation

-- δ_p - pressure perturbation $\quad p = \bar{p} + \delta_p \bar{p}$

The ratio of the (dimensional) pressure and density perturbations is an important property of the fluid (as the equation-of-state that related the mean pressure and density was). It is called the **speed of sound c_s**:

$$c_s = \left(\frac{\delta_p \, \bar{p}}{\delta \, \bar{\rho}} \right)^{1/2}$$

(it will become clear later why this quantity is the velocity of propagation of density waves in the fluid)

For an adiabatic fluid, it can also be computed as:

$$c_s = \left(\frac{\partial p}{\partial \rho} \right)_s^{1/2}$$

-- δu - velocity perturbation

In the case of the homogeneous Universe, the background 4-velocity was $u_\mu = (-a,0)$ (from its normalization) → there was no spatial velocity contribution → the homogeneous fluid was comoving with the expansion.

Now, on the contrary, there is a velocity perturbation and $u_\mu = u_\mu + \delta u_\mu$

with $\delta u_\mu = a\,(-\Phi\ ,v_i + w_i)$

v_i is **the fluid velocity perturbation** - the **peculiar velocity**

w_i comes from the vector metric perturbation

The usual (SV) decomposition defines a scalar part of v_i, such that $v_i = \mathrm{grad}(\theta)$: the **scalar velocity perturbation θ** (also sometimes called v)

The source quantities may also in general include the components of a non-perfect fluid:

-- q_i - energy flux

The energy flux (it is a perturbation there is no need to define a δq, since it was zero in the background) is a velocity vector, usually decomposed in S and V parts.

-- Π_{ij} - anisotropic stress

Anisotropic perturbations in the spatial part of $T_{\mu\nu}$ form the Π_{ij}.
It is decomposed in S (written as second-order derivatives of a scalar σ), V, T

Gauge transformation

To define a metric perturbation, we need both a perturbed and an unperturbed metric → the value of the metric perturbation at (x,t) is the difference between the metric value in the inhomogeneous universe at (x,t) and the metric value that would exist without perturbations at the same point (x,t).

But it is the metric that defines the points (x,t) → the two sets of points (x,t) do not exist in the same space-time → we cannot uniquely say that one point is the same in different metrics → in order to define the perturbation we need to make an identification, a mapping, between points in the 2 metrics → the mapping fixes the gauge.

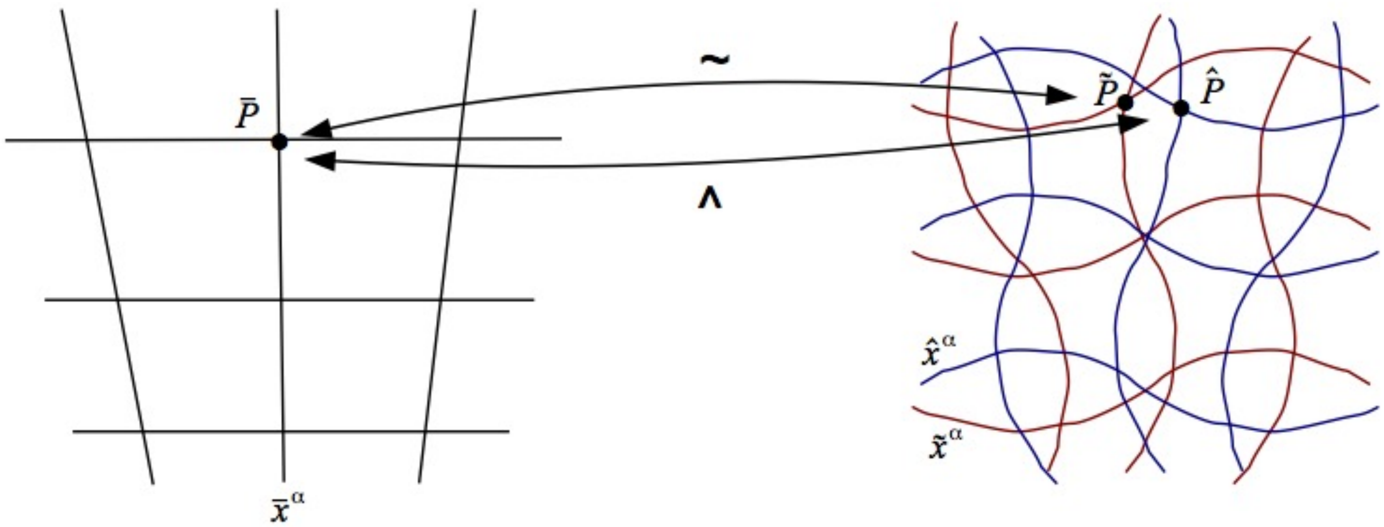
Gauge means a standard, a prescription.

Consider 2 different gauges (mappings), ~ and ^

→ a point in the background metric is identified with 2 different points in the perturbed metric

$$x^{\alpha}(\bar{P}) = \tilde{x}^{\alpha}(\tilde{P}) = \hat{x}^{\alpha}(\hat{P})$$

→ the 2 points have the same coordinates (x,t) in the 2 gauges → a quantity defined at (x,t) in both gauges may be different in the 2 gauges → it is not gauge invariant.



Consider the gauge ~

The 2 'equivalent' points have different coordinates in that gauge → the transformation between the 2 points is the gauge transformation:

$$\tilde{x}^\alpha(\tilde{P}) = \tilde{x}^\alpha(\hat{P}) - \xi^\alpha$$

The transformation is described by the 4-vector $\xi^\mu = (\xi^0, \xi^i)$.

The spatial part may be decomposed as usual in a scalar and a two-component divergence-free vector:

$$\xi\,,\quad \xi_i^{\perp}$$

We can apply this generic gauge transformation to any quantity defined in the space-time.

Transformation of a scalar function:

$$s(\tilde{P}) = s(\hat{P}) + \frac{\partial s}{\partial \hat{x}^\alpha} \underbrace{\left(\hat{x}^\alpha(\tilde{P}) - \hat{x}^\alpha(\hat{P})\right)}_{-\xi^\alpha}$$

(Taylor expansion)

$$\stackrel{(1)}{=} s(\hat{P}) - \frac{\partial \bar{s}}{\partial x^\alpha}\xi^\alpha \stackrel{(2)}{=} s(\hat{P}) - \bar{s}'\xi^0.$$

(1) taking the derivative in the background metric
(2) only time component is needed since the background is isotropic

(where s' is the conformal derivative of s)

In the case the **scalar is a perturbation**: $\tilde{\delta s} \equiv s(\tilde{P}) - \bar{s}(\bar{P})$

$$\tilde{\delta s} = s(\tilde{P}) - \bar{s}(\bar{P}) = s(\hat{P}) - \bar{s}(\bar{P}) - \bar{s}'\xi^0 = \hat{\delta s} - \bar{s}'\xi^0$$

The trivial solution is that any conformal time-invariant scalar is gauge-invariant.

Transformations may also be written for vectors, tensors, and for **vector perturbations**

$$\begin{aligned} \delta\tilde{w}_\mu &= w_{\tilde{\mu}}(\tilde{P}) - \bar{w}_\mu(\bar{P}) \\ &= \delta\hat{w}_\mu - \bar{w}_{\mu,\alpha}\xi^\alpha - \bar{w}_\sigma \xi^\sigma_{,\mu} \end{aligned}$$

and **tensor perturbations**

$$\begin{aligned} \delta\tilde{B}_{\mu\nu} &= B_{\tilde{\mu}\tilde{\nu}}(\tilde{P}) - \bar{B}_{\mu\nu}(\bar{P}) \\ &= \delta\hat{B}_{\mu\nu} - \bar{B}_{\mu\nu,\alpha}\xi^\alpha - \xi^\rho_{,\mu}\bar{B}_{\rho\nu} - \xi^\sigma_{,\nu}\bar{B}_{\mu\sigma} \end{aligned}$$

The total metric perturbation is a tensor perturbation, and this last formula applies.

Using the fact that the background metric is a symmetric and homogeneous tensor, we get the expression for the **gauge transformation of the metric perturbations**:

$$\delta\tilde{g}_{\mu\nu} = \delta\hat{g}_{\mu\nu} - \bar{g}'_{\mu\nu}\xi^0 - 2\xi_{(\mu,\nu)}$$

We can apply this expression to compute the gauge transformation for all metric components. For example, for the (0,0) component:

$$-2a^2\tilde{\phi} = -2a^2\hat{\phi} + 2aa'\xi^0 + 2a^2\xi^{0\prime}$$

$$\Rightarrow \tilde{\phi} = \hat{\phi} - \mathcal{H}\xi^0 - \xi^{0\prime}.$$

The gauge transformations of the 4 scalar components of the metric are:

$$\begin{aligned}
\tilde{\phi} &= \hat{\phi} - \mathcal{H}\xi^0 - \xi^{0\prime}, \\
\tilde{\psi} &= \hat{\psi} + \mathcal{H}\xi^0, \\
\tilde{w} &= \hat{w} + \xi^0 - \xi', \\
\tilde{h} &= \hat{h} - \xi,
\end{aligned}$$

We can also compute the gauge transformations for the **perturbed energy-momentum components**.

For example:

δ scalar perturbation: $$\tilde{\delta} = \frac{\tilde{\delta\rho}}{\bar{\rho}} = \frac{\hat{\delta\rho} - \bar{\rho}'\xi^0}{\bar{\rho}} = \hat{\delta} - \frac{\bar{\rho}'}{\bar{\rho}}\xi^0 = \hat{\delta} + 3\mathcal{H}\xi^0.$$

v_i vector perturbation: $$\tilde{v}_i^{\perp} = \hat{v}_i^{\perp} + \xi_i^{\perp\prime}$$

$$\tilde{v} = \hat{v} + \xi';$$

Fixing the gauge

Defining a particular (arbitrary) ξ fixes the gauge.

The transformation with the 4-vector ξ introduces 4 constraints between the 10 metric perturbations (2S + 2V) → reduces the number of scalar degrees of freedom from 4 to 2, and vector dof from 4 to 2 and keeps the number of tensor dof at 2 → reduces the total degrees of freedom from 10 to 6 → **there are only 6 independent components of the metric perturbations.**

Instead of defining the quadrivector ξ, the gauge can alternatively be fixed by assigning the values of 4 perturbations (2S + 2V).

Some examples of gauges:

Synchronous gauge

w = 0 → no cross terms x,t in the metric → allows to define comoving observers for which x does not change as time goes by (just like it happens for the background RW).

Φ = 0 → all comoving observers (at different x positions) have synchronous time → no gravitational redshift (i.e., no conformal cosmological redshift).

In this gauge the two scalar perturbations remaining are:

ψ and h, that only affect the spatial ii and ij components.

(and there are also 2V and 2T d-o-f remaining)

(Conformal) Newtonian gauge (also called longitudinal gauge)

w = 0 → no cross terms x,t in the metric → allows to define comoving observers for which x does not change as time goes by (just like it happens for the background RW).

h = 0 → spatial perturbations are diagonal → no shear perturbations

In this gauge the metric is defined by ψ and Φ (besides 2V and 2T dof)

Φ gives the gravitational redshift → it is a gravitational potential (taking the limit of GR for weak fields, like in Newtonian gravity, hence the name of this gauge)

ψ is called the curvature potential

From the choice of metric perturbation values in two particular gauges, we can compute the transformation ξ between the two gauges.

For example, the scalar part of the gauge transformation between the synchronous and Newtonian gauge is:

$$\xi^0 = \frac{h'_S + \psi'_S}{2k^2}$$

(Note that here we are working with quantities in Fourier space → the transformation is function of scale.

With this we can compute the **transformations between these gauges** for all quantities:

the **density contrast** transforms between the synchronous and the Newtonian gauge as:

$$\delta_S = \delta_N - \xi^0 \frac{\bar{\rho}'}{\bar{\rho}}$$

the **velocity perturbation** transforms as: $v_S = v_N + \xi'$

Gauge invariance

We see that the metric perturbations are different from gauge to gauge and δ depends on the gauge,

however, observables should be gauge-independent.

The measurement of a power spectrum (or δ value) should not depend on the theoretical choice of the gauge.

Looking at the gauge transformation expression, we see this is indeed the case for large values of k (small scales), and for Universes with metric perturbations that vary slowly → this is the case of sub-Hubble scales (small, intermediate and even large scales in the late universe)

Only for the 'very relativistic universe' is there an ambiguity in observations, i.e. for very large scales or in the early universe.

However, some gauge-invariant combinations can be defined. On very large scales these quantities are the ones that have physical meaning.

An example of gauge-invariant metric quantities are the Bardeen potentials:

$$\Phi \equiv \phi + \frac{1}{a}\left[(w - h')a\right]'$$

$$\Psi \equiv \psi - \mathcal{H}(w - h')$$

Notice that in the Newtonian gauge, these gauge-invariant Bardeen potentials are identical to the scalar perturbations.

An example of a gauge-invariant metric-source quantity is the curvature perturbation:

$$\zeta \equiv \frac{1}{3}\delta + \psi$$

The power spectrum of the curvature perturbation is computed in inflation → it gives the initial condition for the potential power spectrum → and consequently for the matter power spectrum.

An example of a gauge-invariant source quantity is the comoving-gauge density contrast:

$$\Delta = \delta + 3\mathcal{H}v$$

Einstein equations

scalar perturbations

We will consider metric scalar perturbations and derive the **Einstein equations** in the **Newtonian gauge** to linear order

In this case, the perturbed Robertson-Walker metric is:

$$ds^2 = a(\eta)^2 \left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right]$$

Note that there are different sign conventions (+,-) (-,+) (+,+) and different naming conventions found in the literature:

For example,

$$ds^2 = a(\eta)^2 \left[-(1+2\Psi)d\eta^2 + (1+2\Phi)\delta_{ij}dx^i dx^j\right]$$ is used in Dodelson

$$ds^2 = a(\eta)^2 \left[-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right]$$ is used in Liddle & Lyth and Baumann

In order to write the Einstein equations, we need first to compute the following quantities:

Einstein tensor: $$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu$$

Ricci scalar: $$R \equiv R^0_0 + R^i_i$$

Ricci tensor: $$R_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\alpha\mu,\nu} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}$$

Connection: $$\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\lambda}(g_{\lambda\beta,\mu} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\lambda})$$

given our metric: $$g_{\mu\nu} = a^2 \begin{pmatrix} -1-2\Phi & 0 \\ 0 & (1-2\Psi)\delta_{ij} \end{pmatrix}$$

and its inverse: $$g^{\mu\nu} = a^{-2} \begin{pmatrix} -1+2\Phi & 0 \\ 0 & (1+2\Psi)\delta_{ij} \end{pmatrix}$$

We need first to compute the **connections**.

For example, for the term 000 we have,

$$\Gamma^0_{00} = \frac{1}{2} g^{00} g_{00,0} = \frac{1}{2a^2}(-1+2\Phi)[-a^2(1+2\Phi)]' =$$
$$= (1-2\Phi)[\Phi' + \frac{a'}{a}(1+2\Phi)] =$$
$$= \Phi' + \mathcal{H} + 2\mathcal{H}\Phi - 2\Phi\Phi' - 2\mathcal{H}\Phi - 4\mathcal{H}\Phi^2 \simeq \Phi' + \mathcal{H}$$

The results for all terms are:

$$\Gamma^0_{00} = \mathcal{H} + \Phi', \qquad \Gamma^0_{0k} = \Phi_{,k}, \qquad \Gamma^0_{ij} = (\mathcal{H} - 2\mathcal{H}(\Phi+\Psi) + \Psi')\delta_{ij},$$
$$\Gamma^i_{00} = \Phi_{,i}, \qquad \Gamma^i_{0j} = (\mathcal{H} - \Psi')\delta_{ij}, \qquad \Gamma^i_{kl} = -(\Psi_{,l}\delta^i_k + \Psi_{,k}\delta^i_l) + \Psi_{,i}\delta_{kl}.$$

Note the results show **a natural separation between the background RW and perturbation**:

$$\Gamma^{\alpha}_{\beta\gamma} = \bar{\Gamma}^{\alpha}_{\beta\gamma} + \delta\Gamma^{\alpha}_{\beta\gamma}$$

where

$$\bar{\Gamma}^{0}_{00} = \mathcal{H} \qquad \bar{\Gamma}^{0}_{0k} = 0 \qquad \bar{\Gamma}^{0}_{ij} = \mathcal{H}\delta_{ij}$$

$$\bar{\Gamma}^{i}_{00} = 0 \qquad \bar{\Gamma}^{i}_{0j} = \mathcal{H}\delta^{i}_{j} \qquad \bar{\Gamma}^{i}_{kl} = 0$$

and

$$\delta\Gamma^{0}_{00} = \Phi' \qquad \delta\Gamma^{0}_{0k} = \Phi_{,k} \qquad \delta\Gamma^{0}_{ij} = -\left[2\mathcal{H}(\Phi + \Psi) + \Psi'\right]\delta_{ij}$$

$$\delta\Gamma^{i}_{00} = \Phi_{,i} \qquad \delta\Gamma^{i}_{0j} = -\Psi'\delta^{i}_{j} \qquad \delta\Gamma^{i}_{kl} = -\left(\Psi_{,l}\delta^{i}_{k} + \Psi_{,k}\delta^{i}_{l}\right) + \Psi_{,i}\delta_{kl}$$

From this we can compute the **Ricci tensor**

$$R_{\mu\nu} \equiv \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\alpha\mu,\nu} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}$$

(it includes sums over all t,x derivatives and sums of products of two connections)

Note: some useful sums are:

$$\begin{aligned} \Gamma^{\alpha}_{0\alpha} &= 4\frac{a'}{a} + \Phi' - 3\Psi' \\ \Gamma^{\alpha}_{i\alpha} &= \Phi_{,i} - 3\Psi_{,i} \end{aligned}$$

Now, computing for example for 00:

$$R_{00} = \partial_\rho \Gamma^{\rho}_{00} - \partial_0 \Gamma^{\rho}_{0\rho} + \Gamma^{\alpha}_{00}\Gamma^{\rho}_{\alpha\rho} - \Gamma^{\alpha}_{0\rho}\Gamma^{\rho}_{0\alpha}$$

in this case, a term with ρ=0 always cancels out some other term, and so:

$$\begin{aligned} R_{00} &= \partial_i \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{0i} + \Gamma^{\alpha}_{00}\Gamma^{i}_{\alpha i} - \Gamma^{\alpha}_{0i}\Gamma^{i}_{0\alpha} \\ &= \partial_i \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{0i} + \Gamma^{0}_{00}\Gamma^{i}_{0i} + \underbrace{\Gamma^{j}_{00}\Gamma^{i}_{ji}}_{\mathcal{O}(2)} - \underbrace{\Gamma^{0}_{0i}\Gamma^{i}_{00}}_{\mathcal{O}(2)} - \Gamma^{j}_{0i}\Gamma^{i}_{0j} \end{aligned}$$

The results for all terms are:

$$\begin{aligned} R_{00} &= -3\mathcal{H}' + 3\Psi'' + \nabla^2\Phi + 3\mathcal{H}(\Phi' + \Psi') \\ R_{0i} &= 2(\Psi' + \mathcal{H}\Phi)_{,i} \\ R_{ij} &= (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij} \\ &\quad + \left[-\Psi'' + \nabla^2\Psi - \mathcal{H}(\Phi' + 5\Psi') - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)\right]\delta_{ij} \\ &\quad + (\Psi - \Phi)_{,ij} \end{aligned}$$

To compute the Einstein tensor, we also need the Ricci scalar:

$$R \equiv R^0_0 + R^i_i$$

This requires to **raise an index**. Note that this needs to be done using the full metric, we cannot just raise the index of the background and perturbed parts separately:

$$R^\mu_\nu = g^{\mu\alpha}R_{\alpha\nu} = (\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha})(\bar{R}_{\alpha\nu} + \delta R_{\alpha\nu}) = \bar{R}^\mu_\nu + \delta g^{\mu\alpha}\bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta R_{\alpha\nu}$$

(i.e., there are cross-terms)

The results for all terms are:

$$
\begin{aligned}
R^0_0 &= 3a^{-2}\mathcal{H}' + a^{-2}[-3\psi'' - \Delta\Phi + -3\mathcal{H}(\Phi' + \Psi') - 6\mathcal{H}'\Phi], \\
R^0_i &= -2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
R^i_0 &= 2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
R^i_j &= a^{-2}(\mathcal{H}' + 2\mathcal{H}^2)\delta^i_j \\
&\quad + a^{-2}[-\Psi'' + \Delta\Psi - \mathcal{H}(\Phi' + 5\Psi) - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)]\delta_{ij} \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij}.
\end{aligned}
$$

(Note: here the results are given for $R^{\mu}{}_{\nu}$ and not for $R_{\mu\nu}$, hence the a^{-2} factors)

and the **Ricci scalar** is thus:

$$
\begin{aligned}
R &= R^0_0 + R^i_i \\
&= 6a^{-2}(\mathcal{H}' + \mathcal{H}^2) \\
&\quad + a^{-2}\left[-6\Psi'' + 2\nabla^2(2\Psi - \Phi) - 6\mathcal{H}(\Phi' + 3\Psi') - 12(\mathcal{H}' + \mathcal{H}^2)\Phi\right]
\end{aligned}
$$

Finally, the **Einstein tensor** is:

$$
\begin{aligned}
G^0_0 &= -3a^{-2}\mathcal{H}^2 + a^{-2}[2\Delta\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\Phi], \\
G^0_i &= R^0_i = -2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
G^i_0 &= R^i_0 = 2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
G^i_j &= R^i_j - \frac{1}{2}R\delta^i_j \\
&= a^{-2}(-2\mathcal{H}' - \mathcal{H}^2)\delta^i_j \\
&\quad + a^{-2}[2\Psi'' + \Delta(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi]\delta^i_j \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij}.
\end{aligned}
$$

This is the linearized Einstein tensor for the scalar-perturbed Robertson-Walker metric in the conformal Newtonian gauge.

It depends on :

a(t) and its time derivative,

the two metric potentials and their time and spatial derivatives.

Note that the off-diagonal components only have perturbations, while the diagonal components have both perturbations and background terms.

We can now write the Einstein equations $G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu}$

considering the energy-momentum tensor background + perturbations

$$T^{\mu}{}_{\nu} = \bar{T}^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu}$$

$$\bar{T}^{\mu}{}_{\nu} = (\bar{\rho} + \bar{P})\bar{U}^{\mu}\bar{U}_{\nu} - \bar{P}\delta^{\mu}_{\nu}$$

$$\delta T^{\mu}{}_{\nu} = (\delta\rho + \delta P)\bar{U}^{\mu}\bar{U}_{\nu} + (\bar{\rho} + \bar{P})(\delta U^{\mu}\bar{U}_{\nu} + \bar{U}^{\mu}\delta U_{\nu}) - \delta P\,\delta^{\mu}_{\nu} - \Pi^{\mu}{}_{\nu}$$

Remember: the perturbations are density contrast δ, pressure δ_p , peculiar velocity, anisotropy tensor

-- v_i , δ, δ_p are 5 components = 3S+2V

(for scalar perturbations, we just consider the scalar perturbation v associated with the vector v_i → v_i = grad(v))

-- the **traceless anisotropic stress** Π_{ij} accounts for the remaining 5 components = 1S+2V+2T

$$\delta T^i_j = \delta p \delta^i_j + \Sigma_{ij} \equiv \bar{p}\left(\frac{\delta p}{\bar{p}} + \Pi_{ij}\right)$$ (the 3x3 spatial tensor)

There is also the **velocity** 4-vector $u^\mu = \bar{u}^\mu + \delta u^\mu$ with norm -1

in the background: $\bar{u}_\mu \bar{u}^\mu = \bar{g}_{\mu\nu}\bar{u}^\mu \bar{u}^\nu = -a^2(\bar{u}^0)^2 = -1$

the perturbation defines the **peculiar velocity**: $u^i = \bar{u}^i + \delta u^i = \delta u^i \equiv \frac{1}{a} v_i$

Hence, the 4-velocity vector is: $$u^\mu = \frac{1}{a}\begin{pmatrix} 1-\Phi \\ v_{N,i} \end{pmatrix}$$

Note the 0 component does not introduce a new perturbation because of the norm constraint.
The perturbation is the spatial part v_i

In conclusion, **the perturbed part of the energy-momentum tensor** is:

$$\delta T^{\mu}_{\nu} = \begin{bmatrix} -\delta\rho^{N} & -(\bar{\rho}+\bar{p})v^{N}_{,i} \\ (\bar{\rho}+\bar{p})v^{N}_{,i} & \delta p^{N}\delta^{i}_{j} + \bar{p}(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^{2}\Pi) \end{bmatrix}$$

Note that the velocity perturbation does not contribute to the diagonal at linear order because it would contribute with a quadratic term vδ.

We can now write the Einstein equations

showing only linearized perturbations, i.e.,

- no background zero-order terms present

- no higher-order terms present → not valid for non-linear evolution

$$
\begin{aligned}
\delta G^0_0 &= a^{-2}\left[-2\nabla^2\Psi + 6\mathcal{H}(\Psi' + \mathcal{H}\Phi)\right] = -8\pi G\delta\rho^N \\
\delta G^0_i &= -2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = -8\pi G(\bar\rho + \bar p)v^N_{,i} \\
\delta G^i_0 &= 2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 8\pi G(\bar\rho + \bar p)v^N_{,i} \\
\delta G^i_j &= a^{-2}\left[2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi\right]\delta^i_j \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij} = 8\pi G\left[\delta p^N\delta^i_j + \bar p(\Pi_{,ij} - \tfrac{1}{3}\delta_{ij}\nabla^2\Pi)\right] .
\end{aligned}
$$

The ij equations can be separated in diagonal and off-diagonal parts, and **the full set of equations** is,

$$
\begin{aligned}
3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \nabla^2\Psi &= -4\pi G a^2 \delta\rho^N \\
(\Psi' + \mathcal{H}\Phi)_{,i} &= 4\pi G a^2 (\bar{\rho} + \bar{p}) v^N_{,i} \\
\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \tfrac{1}{3}\nabla^2(\Phi - \Psi) &= 4\pi G a^2 \delta p^N \\
(\partial_i\partial_j - \tfrac{1}{3}\delta^i_j\nabla^2)(\Psi - \Phi) &= 8\pi G a^2 \bar{p}(\partial_i\partial_j - \tfrac{1}{3}\delta^i_j\nabla^2)\Pi
\end{aligned}
$$

The equations can also be written in **Fourier space**:

$$
\begin{aligned}
\left(\frac{k}{\mathcal{H}}\right)^2 \Psi &= -\tfrac{3}{2}\left[\delta^N + 3(1+w)\frac{\mathcal{H}}{k} v^N\right] \\
\left(\frac{k}{\mathcal{H}}\right)^2 (\Psi - \Phi) &= 3w\Pi \\
\mathcal{H}^{-1}\Psi' + \Phi &= \tfrac{3}{2}(1+w)\frac{\mathcal{H}}{k} v^N \\
\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}(\Phi' + 2\Psi') + \left(1 + \frac{2\mathcal{H}'}{\mathcal{H}^2}\right)\Phi - \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi - \Psi) &= \tfrac{3}{2}\frac{\delta p^N}{\bar{\rho}},
\end{aligned}
$$

In the case of a perfect fluid ($\Pi_{ij} = 0$) and only scalar fluid perturbations, there are only 4 independent Einstein equations (00, 0i, ii, ij) since all spatial i are identical.

In this case, the 4 **first-order linearized Einstein equations** in the Newtonian gauge reduce to:

$$\nabla^2\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi G a^2 \bar{\rho}\delta$$ "Friedmann / Poisson"

$$\Psi' + \mathcal{H}\Phi = -4\pi G a^2 (\bar{\rho} + \bar{p}) v$$ new equation "velocity"

$$\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p$$ "Raychaudhuri / eq. movement"

$$\Psi - \Phi = 0$$ new equation "anisotropy"

We see that there are 4 fundamental Einstein equations at first-order perturbative level, in contrast with only 2 at background level.

For dark matter (no pressure or pressure perturbations) they can be used to solve for the 4 unknowns: Φ, Ψ, δ, v

We can also write separate **zeroth-order Einstein equations,** i.e., for the homogeneous background.

Since $T_{\mu\nu}$ is the sum of background + matter perturbations and only two of the Einstein tensor components (G_{00} and G_{ll}) are a sum of background + metric perturbations, there are only 2 background Einstein equations.

$$
\begin{aligned}
G^0_0 &= -3a^{-2}\mathcal{H}^2 + a^{-2}[2\Delta\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\Phi], \\
G^0_i &= R^0_i = -2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
G^i_0 &= R^i_0 = 2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
G^i_j &= R^i_j - \frac{1}{2}R\delta^i_j \\
&= a^{-2}(-2\mathcal{H}' - \mathcal{H}^2)\delta^i_j \\
&\quad + a^{-2}[2\Psi'' + \Delta(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi]\delta^i_j \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij}.
\end{aligned}
$$

The two background Einstein equations are

$$\mathcal{H}^2 = \frac{8\pi G}{3}\bar{\rho}a^2,$$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\bar{\rho} + 3p)\, a^2$$

notice that

$$(\mathcal{H}' + \mathcal{H}^2)\, a^{-2} = \frac{\ddot{a}}{a} + H^2$$

$$2\mathcal{H}' = -\mathcal{H}^2(1 + 3w)$$

$$2\mathcal{H}' + \mathcal{H}^2 = -3w\mathcal{H}^2$$

i.e., we recover **Friedmann and Raychaudhuri equations.**

Let us go through the equations one by one.

00 - the Hamiltonian constraint

This equation relates the Laplacian of the potential with the matter density → it is a relativistic Poisson equation.

The two new terms, Ψ' and Φ, function of the potentials, are relativistic corrections to the Newtonian Poisson equation.

The corresponding background equation is the Friedmann equation

→ so Friedmann equation is a kind of Poisson equation, relating the density with gravity (metric) properties.

In the homogeneous case the metric property is the scale factor and not the potential. The potential is a perturbation and does not appear in the homogeneous FRW universe.

The scale factor is related to the "potential of the homogeneous universe", being responsible for the redshift (like the potential is responsible for a gravitational redshift). The potential has dimensions of velocity square → the Hubble flow.

0i - the momentum constraint

This is the peculiar velocity equation.

It has no background counterpart.

Combining equations 00 and 0i, we can cancel out the relativistic corrections and obtain a Poisson equation for the gauge-invariant Δ

$$\nabla^2\Psi = 4\pi G a^2 \bar{\rho}(\delta + 3\mathcal{H}v)$$

that thus defines the GR meaningful "**effective density contrast**".

ii - the pressure constraint (evolution equation of the potential)

This equation involves second-order time derivative of the potential → it is an equation of movement of the potential, describing the evolution of the metric perturbation.

The corresponding background equation is the Raychaudhuri equation → it is the equation of movement for the scale factor.

ij - the anisotropy constraint

This equation tells us that the two Bardeen potentials are equal → it is called the anisotropy equation.

If there is anisotropic stress, the two potentials are no longer equal → in GR, a perfect fluid always induces a metric with equal potentials.

It has no background counterpart.

Let us see a few results of these equations.

Equation 4 (ij): anisotropy equation Ψ

A detection of a difference between the potentials (in the case of a perfect fluid) is a possible **signature of modified gravity**.

This signature is usually parameterized introducing the gravitational slip parameter η

$$\Psi = (1+\eta)\Phi$$

Since there are 2 independent scalar metric perturbations → 2 scalar dof → 2 gravitational potentials in a relativistic theory of gravitation → there is room for a second independent modified gravity signature.
This is usually parameterized by the mass screening parameter Q, or equivalently by an effective gravitational constant G_eff.

This means that G would be different in that theory → it would be equivalent to consider that the same value of the potential is created by a different value of density, through a modified Poisson equation:

$$\nabla^2\Psi = -4\pi G Q a^2 \bar{\rho}\Delta$$

Equation 3 (ii): evolution of the potential Φ

Let us start by introducing the **sound speed** in the equation

$$c_s = \left(\frac{\delta_p \, \bar{p}}{\delta \, \bar{\rho}} \right)^{1/2}$$

We see that the right-hand sides of equations 00 and ii only differ by a factor c_s^2, i.e. → 00 = ii c_s^2

Inserting eq. 00 in eq. ii, and using eq. ij ($\Psi = \Phi$) , we obtain an equation of motion for Φ:

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + \left[2\mathcal{H}' + \mathcal{H}^2(1 + 3c_s^2) - c_s^2\nabla^2\right]\Phi = 0$$

On **small scales**

$$\frac{1}{k} << \mathcal{H}$$

the evolution of the metric perturbation Φ can be approximated by (in the harmonic space)

$$\Phi'' - c_s^2 k^2 \Phi = 0$$

i.e., all terms with H are neglected.

This is a wave equation → Φ oscillates in time, propagating with a velocity given by c_s.

This equation confirms that **the ratio of the pressure to the density perturbations is the velocity of propagation in the fluid.**

On **large scales**

the terms with k are neglected

In the case of a barotropic fluid: p = wρ

$\rightarrow c_s^2 = w$

In the case of an adiabatic fluid: $c_s^2 = \partial p/\partial \rho$

In this case, the evolution of the potential is given by:

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' = 0$$

(since $2\mathcal{H}' = -\mathcal{H}^2(1+3w)$)

This second-order differential equation has 2 solutions:

-- a constant $\rightarrow$ ***the potential remains constant in time***

-- a **decaying solution**

The actual solution Φ (t) depends on the background evolution H(t).

In the late universe when dark energy becomes important, the dominating behaviour is the decaying solution → the potential decreases with time.

That evolution can be used to **test dark energy and modified gravity models** → when CMB photons cross an evolving LSS potential they are blue-shifted (gain energy when entering) and then redshifted (lose energy when leaving). The energy balance is not zero, they gain energy if the potentials decay → their temperature increases with respect to their original temperature → it is the integrated Sachs-Wolfe effect.

The effect is larger on large scales (because photons take longer to cross the larger potentials) → **it is measurable as a change in the amplitude of the CMB power spectrum at large scales.**

Equation 2 (0i): evolution of the peculiar velocity v

Taking now the 0i equation, and inserting the constant potential solution, and the Friedmann equation, the equation for the velocity becomes,

$$v_N = -\frac{2\Phi}{3\mathcal{H}}.$$

In the matter-dominated epoch, the conformal Hubble function decreases as $a^{-1/2}$ → ***the peculiar velocity grows with $a^{1/2}$*** *as dark matter clusters* in the matter-dominated epoch.

Equation 1 (00): evolution of the density contrast δ

Inserting the constant potential solution in the 00 equation (Poisson), and using Friedmann's equation, the equation for the density becomes,

$$\delta_N = -2\Phi - \frac{2k^2}{3\mathcal{H}^2}\Phi$$

Small-scales → the k^2 term dominates.

In the matter-dominated epoch, the conformal Hubble function decreases as $a^{-1/2}$
→ ***the density contrast grows with a***

Large-scales → the constant term dominates.

δ does not grow.
However, on large scales we need to consider the comoving gauge-invariant density contrast Δ. This is the one that enters the relativistic Poisson equation and is the quantity that has physical meaning in a general relativistic covariant framework.

From this Poisson equation, we see that:

- radiation epoch → $\Delta \sim a^{-2} a^{4} \sim a^{2}$

- matter epoch → $\Delta \sim a^{-2} a^{3} \sim a^{1}$

This result can also be found with the mini-universe approach.

Energy conservation equations

Now: the Einstein equations do not contain differential equations for the source perturbations, but only for the metric perturbations.

However, observations measure parameters of the source (not of the metric potential) → **it would be more convenient to study the evolution of δ from a differential equation for δ, defining initial conditions (cosmological parameters) for δ.**

Like it is done for the background, we can obtain more equations by considering the energy conservation of the energy-momentum tensor:

$$\nabla_\mu T^\mu_\nu \;\; = \;\; 0.$$

i.e.,

$$T^\mu_{\nu;\mu} \;=\; T^\mu_{\nu,\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha \;=\; 0$$

At first-order we obtain 2 conservation equations (instead of a single one as was the case for the background)

ν = 0

$$\partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 + \underbrace{\Gamma^\mu_{\mu i} T^i{}_0}_{\mathcal{O}(2)} - \Gamma^0_{00} T^0{}_0 - \underbrace{\Gamma^0_{i0} T^i{}_0}_{\mathcal{O}(2)} - \underbrace{\Gamma^i_{00} T^0{}_i}_{\mathcal{O}(2)} - \Gamma^i_{j0} T^j{}_i = 0$$

This case has a time derivative of T_{00} and a spatial derivative of T_{0i} ,
plus dependence on the potential through the metric (covariant derivative).

Inserting the energy-momentum components and the connection coefficients, the result is an **energy conservation equation**.

Collecting the pure background terms, the result is the zero-order continuity equation, that accounts for the energy conservation in the expanding background:

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{P})$$

The remaining terms are the first-order relativistic continuity equation:

$$\delta' + 3\mathcal{H}(c_s^2 - w)\,\delta + (1+w)(\nabla.v + 3\Phi') = 0$$

note that the divergence of the peculiar velocity is usually denoted $\theta = \nabla.v$

We can compare it with the Newtonian first-order (linearized) comoving continuity equation (for dark matter):

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla.\mathbf{v} = 0$$

For dark matter ($w=0$, $c_s^2 = 0$), the only difference (i.e. the relativistic correction) is the term with the derivative of the potential, that is negligible for slow-varying or constant potentials.

ν = 1

$$\partial_0 T^0{}_i + \partial_j T^j{}_i + \Gamma^\mu_{\mu 0} T^0{}_i + \Gamma^\mu_{\mu j} T^j{}_i - \Gamma^0_{0i} T^0{}_0 - \Gamma^0_{ji} T^j{}_0 - \Gamma^j_{0i} T^0{}_j - \Gamma^j_{ki} T^k{}_j = 0$$

This case has a time derivative of T_{0i} and spatial derivatives, plus dependence on the potential through the metric (covariant derivative).

At background level there is no T_{0i} term and thus there is just one conservation equation.

At perturbative level we get a **momentum conservation equation:**

$$\theta' + \left[\mathcal{H}(1-3w) + \frac{w'}{1+w}\right]\theta = -\nabla^2\left(\frac{c_s^2}{1+w}\delta + \Phi\right)$$

This is also a fundamental equation in fluid dynamics - the **Euler equation** - it is the (acceleration) equation of movement of a Newtonian fluid.

We can compare it with the Newtonian first-order (linearized) comoving Euler equation (for dark matter):

$$\frac{\partial v}{\partial t} + \frac{\dot{a}}{a} v = -\frac{1}{a} \nabla_x \delta\Phi$$

For dark matter ($w=0$, $c_s^2 = 0$, $w'=0$), the Newtonian and relativistic equations are identical.

It tells us that the rate of change of velocity depends on the background expansion, and of the gradients of pressure and gravitational potential ("forces").

Like we saw, it has no counterpart in homogeneous cosmology.

These two fluid evolution equations are not independent of the Einstein equations, but they can be used instead of the two Einstein evolution equations, or in combination with them.

They have the interest of introducing explicitly differential equations for the density contrast and peculiar velocity.

Up to now, the results we found in the relativistic approach are not very different from the ones in the Newtonian approach.

The main differences were:

- the Friedmann equation appears as a Poisson equation (no need to introduce it by hand)

- the Raychadhuri equation appears as an equation for the evolution of the potential (was not part of the set of Newtonian equations)

- the relativistic terms of those equations contain new information that allows us to compute the evolution on large scales, and define a gauge-invariant density contrast

- the continuity and Euler equation appear naturally as before

Perturbed Boltzmann equation

However, the energy-momentum fluid description is not always valid.

Beyond background level, radiation is not well described by a cosmological fluid approach.

The perturbations in the plasma density cannot be described by a coherent fluid with a well-defined velocity → various particle fluxes intersect in the global fluid (multi-streams).

Even for dark matter, in the radiation epoch, the evolution is not accurately computed by using an energy-momentum fluid in the Einstein equations.

The energy-momentum conservation must be studied at the level of particles and not at fluid level, using a kinetic approach (statistical physics) → a transport equation that describes the evolution of a distribution function f(x,p,t) of the cosmological species in the phase space.

The evolution of a distribution function f(x,p,t) is described by the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(\dot{x} f) + \frac{\partial}{\partial p}(\dot{p} f) = \left(\frac{df}{dt}\right)_c$$

or the Vlasov equation if the total derivative of f is conserved (the **collisionless** case):

$$\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \dot{p}\frac{\partial f}{\partial p} = 0$$

The perturbations - density contrast = n(1) / n(0); and velocity v - **are moments of the energy-momentum distribution.**
Remember that the α- order moment of a distribution of a variable, is the integral of the variable over its space weighted by its distribution function.

$$n \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} f$$

(the normalization of f)

$$nv^i \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} f \frac{p\hat{p}^i}{E}$$

(the weighted mean of the velocity)

Since the Boltzmann equation describes the evolution of the distribution f in the phase space → the moments of this equation will be equations that describe the evolution of the moments of particles that follows that distribution → i.e. equations for the evolution of energy density and momentum → i.e. conservation equations.

This description implies a hierarchy of equations, corresponding to the moments of the Boltzmann equation.

For **cold dark matter** the energy and momentum of particles of mass m in the perturbed scalar RW metric, are written as

$$g_{\mu\nu}P^{\mu}P^{\nu} = -m^2 \qquad P^i = \frac{p}{a}(1-\Phi)\hat{p}^i \qquad P^0 = E(1-\Psi)$$

(Notation: here the naming of the potentials is inverted)

The collisionless Boltzmann equation is then:

$$\frac{\partial f}{\partial t} + \frac{p}{aE}\hat{p}^i\frac{\partial f}{\partial x^i} - \left(H\frac{p^2}{E} + \frac{p^2}{E}\dot{\Phi} + \frac{p\hat{p}^i}{a}\partial_i\Psi\right)\frac{\partial f}{\partial E} = 0$$

The **zeroth-order moment** of the collisionless CDM Boltzmann equation for dark matter is found by computing the integral of each term :

$$\dot{n} + \frac{1}{a}\partial_i(nv^i) - (H + \dot{\Phi}) \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\partial f}{\partial E} \frac{p^2}{E} - \frac{1}{a}\partial_i \Psi \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\partial f}{\partial E} p\hat{p}^i = 0$$

Integrating all terms, the result is:

$$\dot{n} + \frac{1}{a}\partial_i(nv^i) + 3(H + \dot{\Phi})n = 0$$

this is the continuity equation

The **first-order moment** of the collisionless CDM Boltzmann equation is its integration in momentum space with its terms multiplied by $p\hat{p}^i/E$

The result is:

$$\frac{\partial v^i}{\partial t} + Hv^i + \frac{1}{a}\partial_i \Psi = 0$$

this is the Euler equation

For cold dark matter, this approach just provided an alternative method that led to the same conservation equations (alternative to using the conservation of the T_{ab} tensor).

However, for perturbations in the radiation component this approach is really needed, since they cannot be described by a fluid.

It is the correct procedure to compute the density perturbations in the radiation-baryonic plasma (needed to compute the CMB power spectrum) or the velocity radiation perturbations (needed to compute dark matter perturbations in multi-fluid coupled equations)

For example, for **photons**, we need to consider the Bose-Einstein distribution function:

$$f(t, p, \boldsymbol{x}, \hat{p}) = \left\{ \exp\left[\frac{p}{T(t)[1 + \Theta(t, \boldsymbol{x}, \hat{p}^i)]} \right] - 1 \right\}^{-1} .$$

where the temperature fluctuations are $\Theta(t, \boldsymbol{x}, \hat{p}^i) \equiv \delta T / T$

The Boltzmann equation leads to the **differential equation for the evolution of the temperature fluctuations**:

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{\boldsymbol{p}} \cdot \boldsymbol{v}_b]$$

For **baryons**, this approach is also needed, but since they are massive particles, the distribution function is different, as well as the relation between energy and momentum.

This allows us to derive the full set of **photon-baryon coupled equations**:
(written in Fourier space)

$$\begin{aligned}
\Theta' &= -\Phi' - ik\mu(\Theta + \Psi) - \tau'_{\rm op}(\Theta_0 - \Theta + \mu v_b)\,, \\
\delta'_b &= -ikv_b - 3\Phi'\,, \\
v'_b &= -\mathcal{H}v_b - ik\Psi + \frac{\tau'_{\rm op}}{R_s}(3i\Theta_1 + v_b)\,,
\end{aligned}$$

The public codes **(CAMB, CLASS)** that compute the linear evolution of cosmological structures for all cosmological species and for a large range of scales and redshift, implement this approach → they solve the system of Einstein-Boltzmann differential equations (in a certain gauge).

Other types of perturbations

Up to know we focused on scalar perturbations.

However, remember that there are also vector and tensor perturbations, and a possible total of 10 Einstein equations.

Vector perturbations

There are 4 V perturbations in the metric and 2 V components of the transformation vector ξ (the 2 vector components $\xi_i^{\perp}$) → the choice of gauge fixes 2 V components of the metric.

This can be done, for example, by setting the vector part of h to 0,
(in addition to fixing 2 scalar components, for example w = h = 0)

There remains 2 V components of the metric (the vector parts of w).

In the Einstein equations, there are 3 equations that involve the vector metric perturbations and the vector source perturbations.

Those 3 equations are:

$$\begin{aligned}
(v_i^{\perp} + w_i^{\perp})' + \mathcal{H}(v_i^{\perp} + w_i^{\perp}) &= 0, \\
\frac{1}{2}\Delta w_i^{\perp} &= 8\pi G a^2 \bar{\rho}(v_i^{\perp} + w_i^{\perp}), \\
\left(w_{(i,j)}^{\perp}\right)' + 2\mathcal{H} w_{(i,j)}^{\perp} &= 0.
\end{aligned}$$

The solution, from the first equation is:

$$v_i^{\perp} + w_i^{\perp} \sim a^{-1}$$

This shows that the vector perturbations decay with time.

The vector part of initial velocity perturbations eventually disappear, and they are not relevant in the standard cosmological model.

Tensor perturbations

There are 2 T perturbations in the metric and no T components of the transformation vector ξ → **tensor perturbations are gauge-invariant by construction** → no gauge fixing needed.

Even if the energy-momentum tensor has no tensor part (no anisotropic stress) there exists still one equation in the Einstein - energy conservation system that involves only tensor metric perturbations

(in fact 2 equations, since there are 2 T components) → these 2 components may also be written as a polarization vector (a polar vector).

The equations are:

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij} = 0$$

This is a second-order differential equation in time and space: a wave equation, also containing a first-order derivative term (a friction term, known as the Hubble drag).

The solution is:

$$h_{ij} \sim e^{i(\omega\tau + \mathbf{k}\cdot\mathbf{x})}$$

This means that the tensor perturbations evolve in time and space in a coherent way, as a propagating wave.

Even with no sources, initial tensor metric perturbations do not vanish and propagate as a wave.
We can say it is an **intrinsic property of GR** → these are the gravitational waves.

It seems a more fundamental property than gravity being attractive, because attraction depends on the source → with no initial sources (δ), there would be no structure formation, but there would still exist gravitational waves.

The amplitude of the wave does not remain constant, it decreases in time due to the Hubble drag term.

Remember that **inflation** sets the **initial conditions** for scalar and tensor metric perturbations:

For scalar perturbations → sets the slope of the primordial power spectrum of the curvature potential (the scalar index n_s) → sets the slope of the primordial matter power spectrum (through Poisson equation).

For tensor perturbations → sets the slope of the primordial power spectrum of tensor perturbations (the tensor index n_t) → no equivalence in a source power spectrum.

These are the **primordial gravitational waves**.

Local interactions of strong gravity can produce **secondary gravitational waves**
→ produced by periodic movement of compact objects: black holes, neutron stars binaries, etc.
(These are the ones that have been detected, not related to cosmology).

Observationally, there are two main **signatures of (cosmological) primordial gravitational waves** that are being explored:

- The metric at a location changes as the wave passes → produces a periodical change in the size (or distance) of objects

- GW polarize the CMB photons → could be detected in the CMB polarization power spectra.

Being a fundamental property of gravitation, GW can also be used to test modified gravity. Some theories of gravity may have a different number of tensor modes → different types of polarization in their gravitational waves.