Jeans Instability and Linear Perturbation Theory at Horizon Re-Entry

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Primordial Universe

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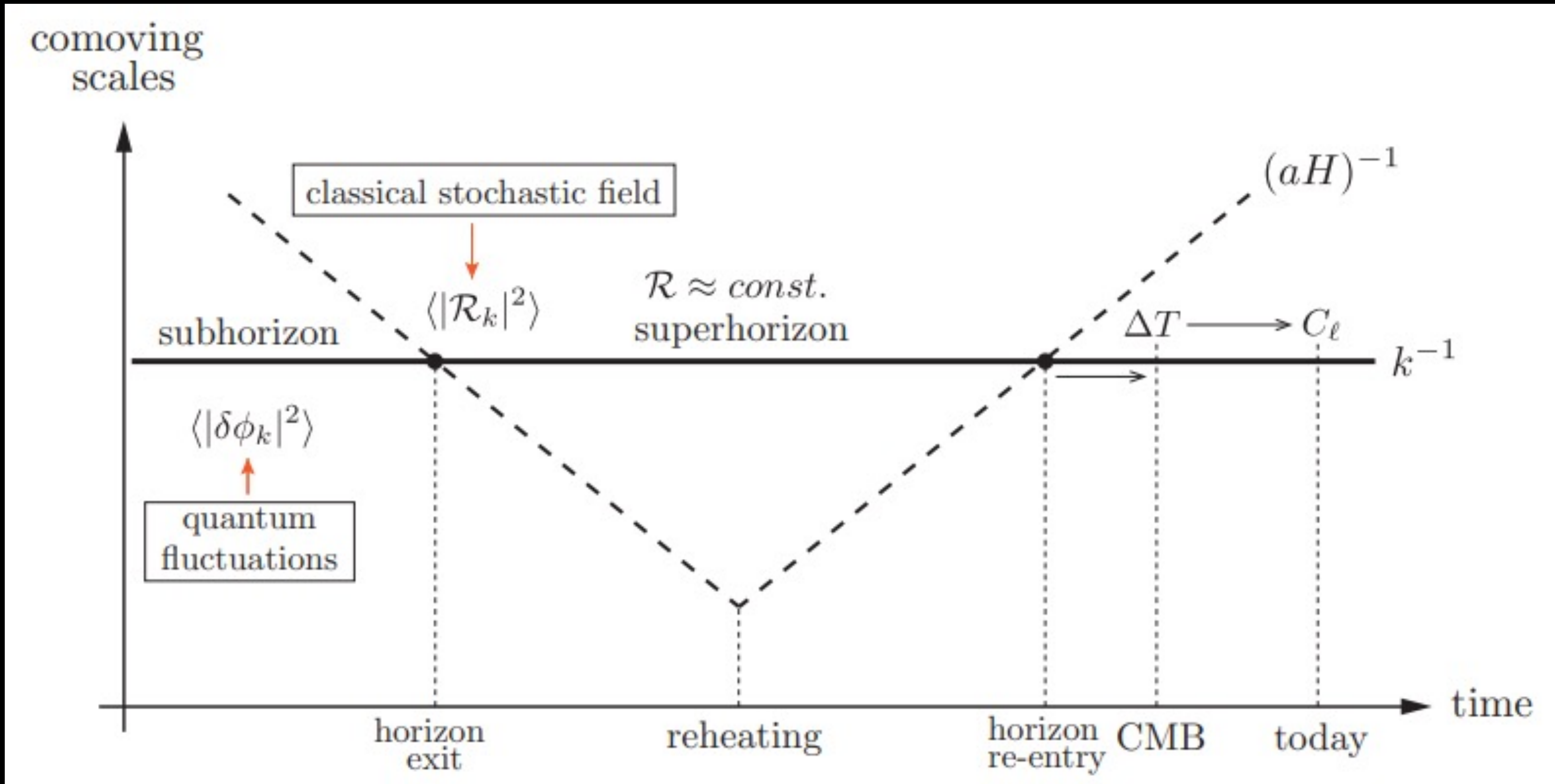
Inflation and the origin of Perturbations



The (Perturbed) Fluid Equations

- Subhorizon and non-relativistic colisional fluid in causal Interaction: Described by ρ(**r**,t), **v**(**r**,t), P(**r**,t), Φ(**r**,t), at constant Entropy.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right) \vec{v} + \frac{\vec{\nabla} P}{\rho} + \vec{\nabla} \Phi = 0$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\frac{\partial}{\partial t}[\rho_0(1+\delta)] + \vec{\nabla} \cdot [(\rho_0(1+\delta)(\vec{v}_0 + \delta\vec{v})] = 0$$

$$\frac{\partial}{\partial t}(\vec{v}_0 + \delta\vec{v}) + (\vec{v}_0 + \delta\vec{v}) \cdot \vec{\nabla}(\vec{v}_0 + \delta\vec{v}) + \frac{\vec{\nabla}(P_0 + \delta P)}{\rho_0(1+\delta)} + \vec{\nabla}(\Phi_0 + \delta\Phi) = 0$$

$$\nabla^2(\Phi_0 + \delta\Phi) = 4\pi G \rho_0 (1+\delta)$$

$$\psi^i(\vec{r}, t) = \psi^i_0(t) + \delta\psi^i(\vec{r}, t), \quad \delta\psi^i << \psi^i_0$$

$$\left(\frac{\partial P}{\partial \rho}\right)_S = c_s^2 \longrightarrow \delta P = c_s^2 \delta\rho = c_s^2 \rho_0 \delta$$

1) Static Space without Gravity

- Neglecting the effects of gravity, ∇(δΦ) → 0, we get a wave equation, whose solution is that of a **Constant Amplitude Plane Wave**.

$$\rho_0 \frac{\partial \delta}{\partial t} + \rho_0 \vec{\nabla} \cdot (\delta \vec{v}) = 0$$

$$\frac{\partial}{\partial t}(\delta \vec{v}) + c_s^2 \vec{\nabla} \delta = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta = 0$$

$$\delta = A e^{i(\omega t - \vec{k} \cdot \vec{r})}, \qquad \omega = \pm c_s k$$

2) Static Space with Gravity

- Considering gravity, we get a wave equation with a source term, admiting a plane wave as a solution, but with an ω that can be real or imaginary → allows **exponential growth of perturbations above Jeans size-scales**.

$$\rho_0 \frac{\partial \delta}{\partial t} + \rho_0 \vec{\nabla} \cdot (\delta \vec{v}) = 0$$

$$\frac{\partial}{\partial t}(\delta \vec{v}) + c_s^2 \vec{\nabla} \delta + \vec{\nabla}(\delta \Phi) = 0$$

$$\nabla^2(\delta \Phi) = 4\pi G \rho_0 \delta$$

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta = 4\pi G \rho_0 \delta$$

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0, \text{ when } \omega = 0 \longrightarrow k_J = \frac{\sqrt{4\pi G \rho_0}}{c_s}$$

$$\lambda_J = \frac{2\pi}{k_J} = c_s \sqrt{\frac{\pi}{G\rho_0}}; \text{ For } \lambda > \lambda_J \text{ , } \delta \propto e^{i\omega t}$$

3) Expanding Space with Gravity

- Using comoving coordinates, in a matter dominated universe, perturbations evolve as damped oscilations (due to the expanding frictional term) below the Jeans length and collapse as power laws above it.

$$\vec{r} = a(t)\vec{x}\,, \quad \vec{v} = H\vec{r} + \vec{u}$$

$$\vec{\nabla}_{\vec{r}} = a^{-1}\vec{\nabla}_{\vec{x}}\,, \left(\frac{\partial}{\partial t}\right)_{\vec{r}} = \left(\frac{\partial}{\partial t}\right)_{\vec{x}} + H\vec{x}\cdot\vec{\nabla}_{\vec{x}}$$

$$\dot{\delta} = -\frac{1}{a}\vec{\nabla}\cdot\vec{u}$$

$$\dot{\vec{u}} + H\vec{u} = -\frac{c_s^2\vec{\nabla}(\delta)}{a} - \frac{1}{a}\vec{\nabla}(\delta\Phi)$$

$$\nabla^2(\delta\Phi) = 4\pi G\rho_0 a^2\delta$$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta = 4\pi G\rho_0\delta$$

$$\omega^2 - 2Hi\omega = \frac{c_s^2}{a^2}k^2 - 4\pi G\rho_0$$

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0 \longrightarrow \delta \propto t^{2/3} \propto a$$

4) The case for (Cold) Dark Matter

- It can be proven, using the Liouville Equation, that a collisionless fluid such as Dark Matter (DM) follow the same dynamics as a collisional fluid, albeit with an averaged velocity of DM particles replacing c_s.

$$\frac{\partial f}{\partial t} + \vec{\nabla}_{\vec{r}} \cdot (f\vec{v}) + \vec{\nabla}_{\vec{v}} \cdot (f\dot{\vec{v}}) = 0$$

$$\nabla^2(\delta\Phi) = 4\pi G m \int \delta f dv$$

$$\lambda_J = \frac{2\pi}{k_J} = 2\pi\sqrt{\frac{\int f dv^3}{4\pi G\rho \int v^{-2} f dv^3}} = v_* \sqrt{\frac{\pi}{G\rho}},$$

$$v_*^{-2} = \frac{\int v^{-2} f dv^3}{\int f dv^3} = \langle v^{-2} \rangle$$

- Consequently Cold (low velocity) Dark Matter easily enters Jeans Collapse at subhorizon scales.

SuperHorizon Perturbations

- What happens to Perturbations before their scale enters the particle horizon?

$$\Phi'' + 3(1+w)\mathcal{H}\Phi' + wk^2\Phi = 0$$

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \rho_0 \delta$$

In superhorizon conditions this leads to $\Phi(\tau) \approx$Cte, and δ directly proportional to Φ.

$$\mathcal{R} = -\frac{5+3w}{3+3w}\Phi$$

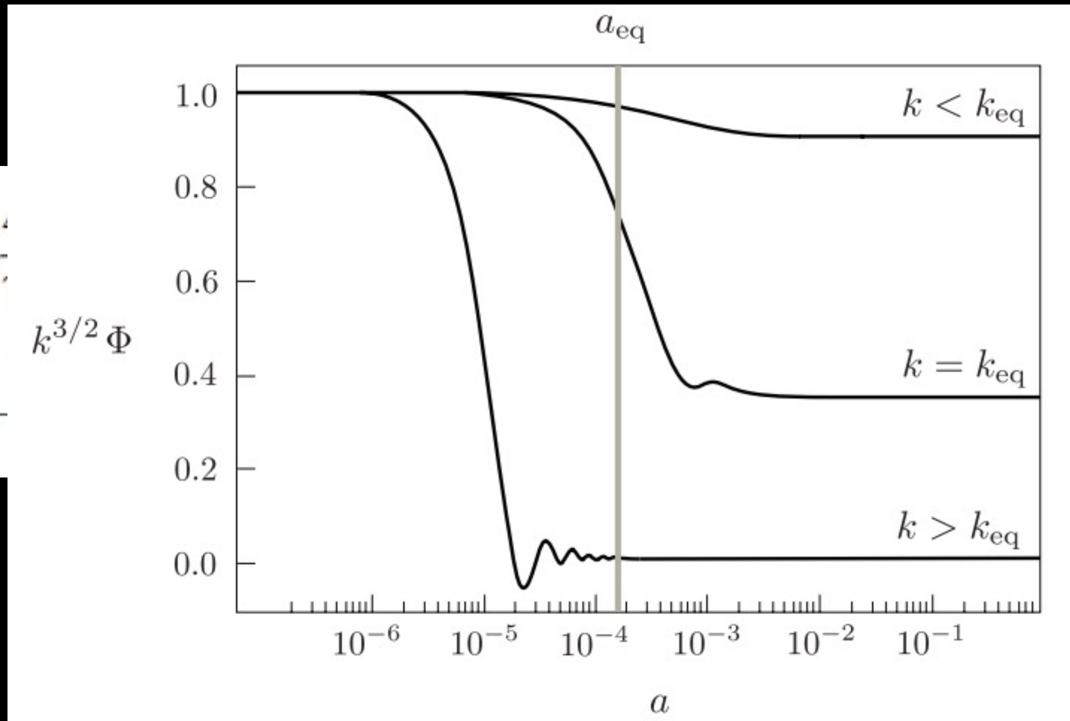
$$\mathcal{R}_{RD} = \mathcal{R}_{MD} \longrightarrow \Phi_{MD} = \frac{9}{10}\Phi_{RD}$$

During the transition from Radiation Domination to Matter Domination, the constant superhorizon potential perturbation drops slightly.

Horizon Re-Entry of Gravitational Perturbations

- Computing now both cases, by solving the motion equation, Φ(τ) oscillates with amplitude decreasing with τ^2 at RD and stays constant during MD.

$$\Phi_k(\tau) \propto \frac{\cos\left(\frac{k\tau}{\sqrt{3}}\right)}{(k\tau)^2}$$



Horizon Re-Entry of Radiation Field Perturbations

- Replacing Φ(τ) into the Poisson Equation during RD we get an oscillatory solution

$$-k^2\Phi - \frac{3}{\tau}\Phi' - \frac{3}{\tau^2}\Phi = \frac{3}{2\tau^2}\delta_r \longrightarrow \delta_r = -\frac{2}{3}k^2\tau^2\Phi - 2\tau\Phi' - 2\Phi$$

$$\delta_r \approx -\frac{2}{3}(k\tau)^2\Phi \propto \cos\left(\frac{1}{\sqrt{3}}k\tau\right), \quad \text{a solution to} \quad \delta_r'' - \frac{1}{3}\nabla^2\delta_r = 0$$

Horizon Re-Entry of Dark Matter Perturbations

- While radiation dominates (y << 1), one obtains the Mészáros equation, leading to a stagnant growth of matter perturbations, at $\delta_m \approx \ln(y) \approx \ln(a)$.

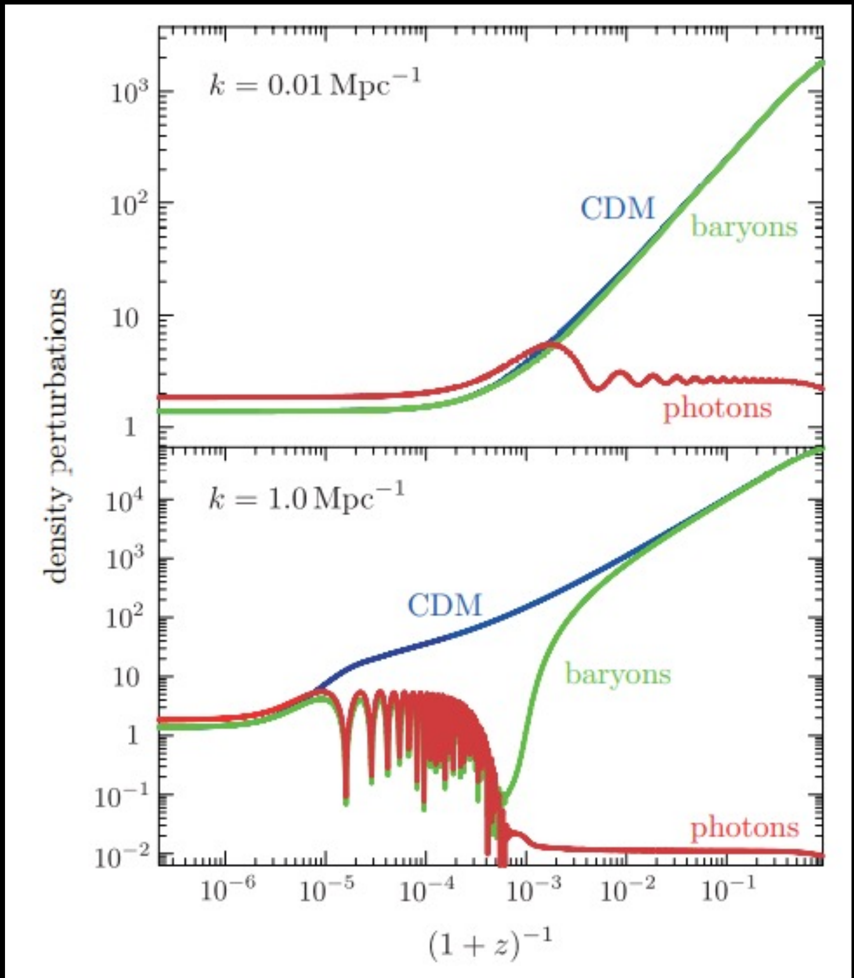
$$\delta_m'' + \mathcal{H}\delta_m' - 4\pi G a^2 \rho_m \delta_m \approx 0$$

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} + \frac{3}{2y(1+y)}\delta_m \approx 0$$

- When matter dominates (y >> 1), one obtains the aforementioned linear growth of perturbation contrast density, δ_m.
- For Late Times… accelerated expansion stops growth:

$$\delta_m'' + \mathcal{H}\delta_m' \approx 0 \longrightarrow \delta_m \approx Const.$$

Horizon Re-Entry of Baryon Perturbations



Before baryons decouple from the radiation fluid, subhorizon baryon fluctuations follow the radiation oscillations – the origin of the Baryonic Acoustic Oscillations.

After decoupling, subhorizon Baryon fluctuations quickly couple to those of Cold Dark Matter, allowing the formation of baryonic structures such as galaxies.

Conclusion – The origin the Universe's structures