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Kaluza-Klein Theory

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Kaluza's Theory

- Kaluza unified gravity with electromagnetism in a theory of five dimensions
- All the derivatives with respect to the fifth dimension vanish (Kaluza's cylinder condition)
- General relativity, when interpreted as a 5D theory in vacuum, contained 4D general relativity in the presence of an electromagnetic field together with Maxwell's laws of electromagnetism

Figure: Theodor Kaluza.

Kaluza's Theory

There are three important aspects of Kaluza's theory:

1. Nature can be explained through pure geometry
2. The higher-dimensional theory is a minimal extension of general relativity
3. Physics only depends on the first four coordinates (cylinder condition) and no mechanism is suggested to explain why it doesn't depend on the extra ones

Solutions for the fifth dimension:

- Compactified theory: Kaluza-Klein theory
- Projective geometry
- Noncompactified theories

Notation

- Capital Latin indices A,B, ... run over 0,1,2,3,4
- Greek indices α,β, ... run over 0,1,2,3
- Five dimensional quantities are denoted by hats
- The fifth coordinate is denotes by $y = x^4$

Kaluza's Theory

The Einstein equations in 5D with no 5D energy-momentum tensor are:

$$\hat{G}_{AB} = 0 \tag{1}$$

or, equivalently:

$$\hat{R}_{AB} = 0 \tag{2}$$

with:

Einstein's tensor: $\hat{G}_{AB} \equiv \hat{R}_{AB} - \hat{R}\hat{g}_{AB}/2$

Ricci tensor: $\hat{R}_{AB}$

Ricci scalar: $\hat{R} = \hat{g}_{AB}\hat{R}^{AB}$

Five-dimensional metric tensor: $\hat{g}_{AB}$

Kaluza's Theory

The previous equations can also by derived by:

$$S = -\frac{1}{16\pi\hat{G}}\int \hat{R}\sqrt{-\hat{g}}d^4xdy \tag{3}$$

The five-dimensional Ricci tensor and Christoffel symbols are:

$$\begin{aligned}
\hat{R}_{AB} &= \partial_C\hat{\Gamma}^C_{AB} - \partial_B\hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC} \\
\hat{\Gamma}^C_{AB} &= \frac{1}{2}\hat{g}^{CD}\left(\partial_A\hat{g}_{DB} + \partial_B\hat{g}_{DA} - \partial_D\hat{g}_{AB}\right)
\end{aligned} \tag{4}$$

Kaluza's Theory

The physics depends now on the choice of the five-dimensional metric.

We make the following ansatz:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\varphi^2 A_\alpha A_\beta & \kappa\varphi^2 A_\alpha \\ \kappa\varphi^2 A_\beta & \varphi^2 \end{pmatrix} \tag{5}$$

Where $g_{\alpha\beta}$ is the 4D metric tensor, A_α is the electromagnetic potential, φ is a scalar field and κ is a multiplicative factor.

Kaluza's Theory

Applying the cylinder condition one obtains:

$$
\begin{aligned}
G_{\alpha\beta} &= \frac{\kappa^2\varphi^2}{2} T^{EM}_{\alpha\beta} - \frac{1}{\varphi}\left[\nabla_\alpha\left(\partial_\beta\varphi\right) - g_{\alpha\beta}\Box\varphi\right] \\
\nabla^\alpha F_{\alpha\beta} &= -3\frac{\partial^\alpha\varphi}{\varphi} F_{\alpha\beta} \\
\Box\varphi &= \frac{\kappa^2\varphi^3}{4} F_{\alpha\beta}F^{\alpha\beta}
\end{aligned}
\tag{6}
$$

Important result

4D matter arises purely from the geometry of empty 5D spacetime.

Compactified Theory

Klein assumed that the fifth coordinate was a lengthlike one and assigned it two properties:

1. Circular topology (S^1)
2. Small scale

Figure: Oskar Klein.

Compactified Theory

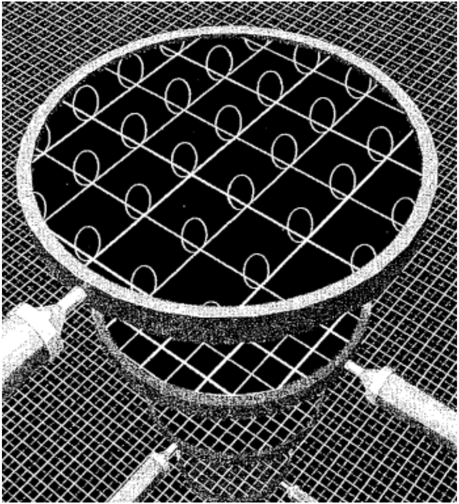

Figure: On very small scales space has an extra circular dimension tacked on to each familiar point.

Compactified Theory

Under property (1) any quantity $f(x,y)$ becomes periodic, $f(x,y) = f(x, y + 2\pi r)$, where r is the "radius" of the fifth dimension.

All the fields can be Fourier-expanded:

$$\begin{aligned} g_{\alpha\beta}(x,y) &= \sum_{n=-\infty}^{n=\infty} g_{\alpha\beta}^{(n)}(x) e^{iny/r} \\ A_{\alpha}(x,y) &= \sum_{n=-\infty}^{n=\infty} A_{\alpha}^{(n)}(x) e^{iny/r} \\ \varphi(x,y) &= \sum_{n=-\infty}^{n=\infty} \varphi^{(n)} e^{iny/r} \end{aligned} \tag{7}$$

Modes carry momentum $\propto |n|/r$, if r is small enough only $n = 0$ contributes.
Theoretical physicists set $r = l_{pl} \sim 10^{-35} m$

Compactified Theory

By introducing a 5D massless scalar field $\hat{\psi}(x, y)$ on the action one gets:

$$S_{\hat{\psi}} = -\int d^4x dy \sqrt{-\hat{g}} \partial^A \hat{\psi} \partial_A \hat{\psi} \tag{8}$$

Expanding the field results in:

$$S_{\hat{\psi}} = -\left(\int dy\right) \sum_n \int d^4x \sqrt{-g} \left[\left(\partial^\alpha + \frac{in\kappa A^\alpha}{r}\right) \hat{\psi}^{(n)} \left(\partial_\alpha + \frac{in\kappa A_\alpha}{r}\right) \hat{\psi}^{(n)} - \frac{n^2}{\varphi r^2} \hat{\psi}^{(n)2} \right] \tag{9}$$

First term: covariant derivative
Second term: mass term

Compactified Theory

- Comparison with covariant derivative from QED leads to:

$$q_n = \frac{n\kappa}{r}\left(\varphi \int dy\right)^{-1/2} = \frac{n\sqrt{16\pi G}}{r\sqrt{\varphi}} \tag{10}$$

- Masses of the scalar modes are:

$$m_n = \frac{|n|}{r\sqrt{\varphi}} \quad . \tag{11}$$

If $r\sqrt{\varphi} \sim l_{pl}$ then $m_1 = l_{pl}^{-1} = m_{pl} \sim 10^{19} GeV \neq 0.5 MeV$.

Compactified Theory

The compactified theory is successful in predicting:

- Quantization of the charge

However, it presents this problems:

- Explicit addition of matter
- Quantized mass
- The mass predicted for the lowest Fourier mode is of the Planck's order.

Noncompactified Theory

In the noncompactified theory we consider the following:

- The extra dimension is real
- Relax cylindricity condition: The nature is only approximately independent of the fifth coordinate

It raises questions like:

- Why nature should be so nearly cylindrical?
- Why the effects of the fifth dimension have not been noticed so far?

Noncompactified Theory

We begin with the same 5D metric as before, but choose coordinates such that A_α vanish.

We include the factor ε to allow timelike and spacelike signature for the fifth coordinate.

The 5D metric tensor is:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \varepsilon\phi^2 \end{pmatrix} \tag{12}$$

Noncompactified Theory

By using the same definitions of the five-dimensional Christoffel symbols and Ricci tensor, but now keeping the derivatives with respect to the fifth coordinate we get these final results:

$$R_{\alpha\beta} = \frac{\nabla_\beta \left(\partial_\alpha \varphi\right)}{\varphi} - \frac{\varepsilon}{2\varphi^2}\left(\frac{\partial_4 \varphi \partial_4 g_{\alpha\beta}}{\varphi} - \partial_4\left(\partial_4 g_{\alpha\beta}\right) + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta}\partial_4 g_{\gamma\delta}\partial_4 g_{\alpha\beta}}{2}\right) \tag{13}$$

$$\nabla_\beta P_\alpha^\beta = 0 \quad ,\text{with} \quad P_\alpha^\beta \equiv \frac{1}{2\sqrt{\hat{g}_{44}}}\left(g^{\beta\gamma}\partial_4 g_{\gamma\alpha} - \delta_\alpha^\beta g^{\gamma\epsilon}\partial_4 g_{\gamma\epsilon}\right) \tag{14}$$

$$\varepsilon\varphi\Box\varphi = -\frac{\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{4} - \frac{g^{\alpha\beta}\partial_4\left(\partial_4 g_{\alpha\beta}\right)}{2} + \frac{\partial_4\varphi g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{2\varphi} \tag{15}$$

Noncompactified Theory

Using equation (13) and inserting it into in the usual Einstein equation, $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$, one finds:

$$8\pi G T_{\alpha\beta} = \frac{\nabla_\beta (\partial_\alpha \varphi)}{\varphi} - \frac{\varepsilon}{2\varphi^2} \left[\frac{\partial_4 \varphi \partial_4 g_{\alpha\beta}}{\varphi} - \partial_4 (\partial_4 g_{\alpha\beta}) + g^{\gamma\delta} \partial_4 g_{\alpha\gamma} \partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta} \partial_4 g_{\gamma\delta} \partial_4 g_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4} \left(\partial_4 g^{\gamma\delta} \partial_4 g_{\gamma\delta} + \left(g^{\gamma\delta} \partial_4 g_{\gamma\delta} \right)^2 \right) \right] \tag{16}$$

The induced-matter interpretation

The induced matter described by $T_{\alpha\beta}$ is a manifestation of pure geometry in higher-dimensional world.

Noncompactified Theory

Equation (14):

$$\nabla_\beta P_\alpha^\beta = 0 \quad \text{, with} \quad P_\alpha^\beta \equiv \frac{1}{2\sqrt{\hat{g}_{44}}} \left(g^{\beta\gamma} \partial_4 g_{\gamma\alpha} - \delta_\alpha^\beta g^{\gamma\epsilon} \partial_4 g_{\gamma\epsilon} \right) \qquad (17)$$

- Interpretation is not clear
- Could be related to a known conserved quantity
- If $P_{\alpha\beta} = k \left(m_i v_\alpha v_\beta + m_g g_{\alpha\beta} \right)$ then $\nabla_\beta P_\alpha^\beta = 0$ is a 4D geodesic equation

Noncompactified Theory

Equation (15):

$$\varepsilon\varphi\Box\varphi = -\frac{\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{4} - \frac{g^{\alpha\beta}\partial_4\left(\partial_4 g_{\alpha\beta}\right)}{2} + \frac{\partial_4\varphi g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{2\varphi} \tag{18}$$

- Takes the form of a scalar wave equation for φ
- Interpretation is not clear
- With appopriate definition of the particle mass it can be identified with Klein-Gordon equation: $\Box\varphi = m^2\varphi$

Noncompactified Theory

The noncompactified theory is successful in:

- giving the energy-momentum tensor from the geometry
- giving testable predictions

However, it presents this problems:

- physical quantities depend on the 5D coordinate-system
- interpretation is not clear

Cosmology

- Noncompactified cosmology, homogeneous and isotropic nonstatic case

The 5D line element for the flat, isotropic and homogeneous universe is:

$$d\hat{s}^2 = e^{\nu} dt^2 - e^{\omega} \left(dr^2 + r^2 d\Omega^2 \right) + \varepsilon e^{\mu} d\psi^2 \qquad (19)$$

Where ε serves the same function as before, t, r, θ and φ have their usual meanings, ψ is the fifth coordinate and ν,ω and μ are arbitrary functions of r, t, ψ.

Cosmology

If one assumes:

- ν,ω and μ are separable functions of t, r and ψ
- $\varepsilon = -1$
- The induced matter takes the form of a perfect fluid: $T^{\alpha}_{\beta} = (\rho + p)u^{\alpha}u_{\beta} - p\delta^{\alpha}_{\beta}$

We obtain the following expressions for density and pressure:

$$\rho = \frac{3}{8\pi G\alpha^2\psi^2 t^2} \quad , \quad p = \left(\frac{2\alpha}{3} - 1\right)\rho \tag{20}$$

Equations of state

- $\alpha = 2 \Rightarrow (p = \rho/3) \Rightarrow$ Radiation-dominated universe
- $\alpha = 3/2 \Rightarrow p = 0 \Rightarrow$ Dust-filled universe
- $0 < \alpha < 1 \Rightarrow$ Inflationary universe

Cosmology

From the previous assumptions we can also obtain the following conclusion:

- Comoving objects satisfy the 5D geodesic equation exactly the same way as in 4D theory
- Noncomoving objects don't satisfy the 4D geodesic equation
- Therefore, galaxies with large peculiar velocities will not necessarily travel along 4D geodesics

Cosmology

The 5D line element for a curved, homogeneous and isotropic universe is:

$$d\hat{s}^2 = e^\nu dt^2 - e^\omega \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - e^\mu d\psi^2 \tag{21}$$

If we assume that the metric coefficients have the form:

$$e^\nu \equiv L^2(t - \lambda\psi) \quad , \quad e^\omega \equiv M^2(t - \lambda\psi) \quad , \quad e^\mu \equiv N^2(t - \lambda\psi) \quad , \tag{22}$$

where L,M and N are wavelike functions of the argument $(t - \lambda\psi)$, with λ acting as a wave number, then we obtain for the perfect fluid:

$$\rho = \frac{3\zeta^2\lambda^2}{8\pi L^{3+3\gamma}} \quad , \quad p = \gamma\rho \quad , \tag{23}$$

Equations of state

- $\gamma = 0 \Rightarrow$ Matter-dominated era
- $\gamma = 1/3 \Rightarrow$ Radiation-dominated era

Cosmology

Equations (22) changes the metric to:

$$d\hat{s}^2 = \frac{1}{L^{1+3\gamma}}dt^2 - L^2 d\chi^2 - \left(\frac{\lambda^2}{L^{1+3\gamma}} - \frac{k}{\zeta^2}\right) d\psi^2 \quad , \tag{24}$$

Where $L(t - \lambda\psi)$ plays the role of the cosmological scale factor.

Consequences

- Observers with different values of ψ would disagree on the age of the universe
- We could constrain the theory using observational data on the age spread of objects such as globular clusters

Thank you!