Density Perturbations from Inflation

Fábio Carmo Nº51334

Universo Primitivo

Horizon Problem

FLRW metric $ds^2 = dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)d\Omega^2\right]$

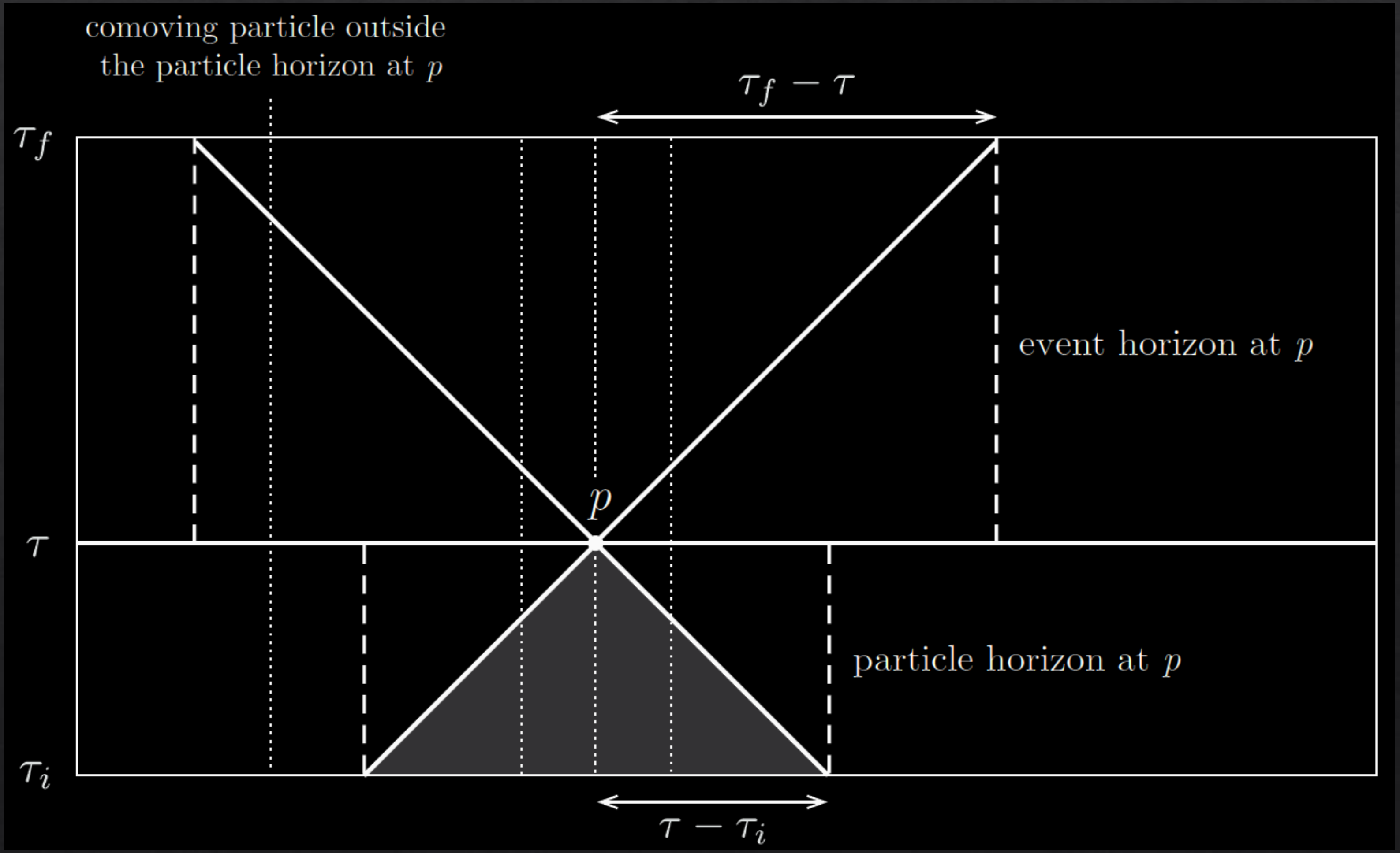
$$S_k(\chi) = \begin{cases} \sinh\chi & k = -1 \\ \chi & k = 0 \\ \sin\chi & k = +1 \end{cases}$$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ $\qquad d\tau = \dfrac{dt}{a(t)}$

$\Longrightarrow$ $ds^2 = a^2(\tau)[d\tau^2 - d\chi^2]$

$\Delta\chi(\tau) = \pm\Delta\tau$

Horizon Problem



Horizon Problem

Particle horizon

$$\chi_{ph} = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a} = \int_{a_i}^{a} \frac{da}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1} \; d\ln a$$

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}$$

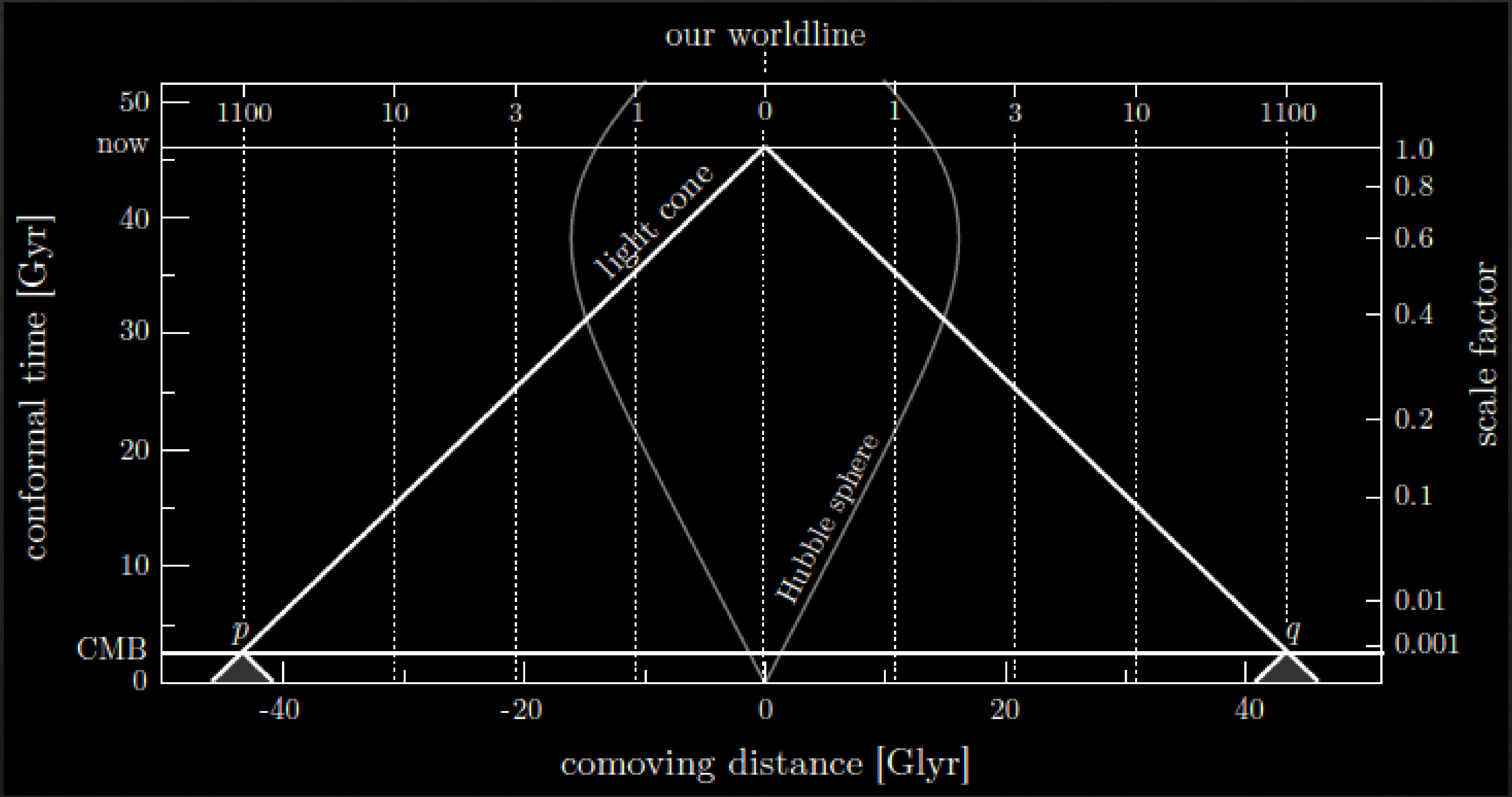
$\dot{a} > 0 \quad 1 + 3w > 0$ →

$$\frac{d(aH)^{-1}}{dt} = H_0^{-1} \frac{1}{2}(1+3w) a^{-\frac{1}{2}(1+3w)} \dot{a} > 0$$

$$\tau_i = \frac{2H_0^{-1}}{(1+3w)} a_i^{\frac{1}{2}(1+3w)}$$

$a_i \to 0$, $w > -\frac{1}{3}$ → 0

Horizon Problem



Horizon Problem

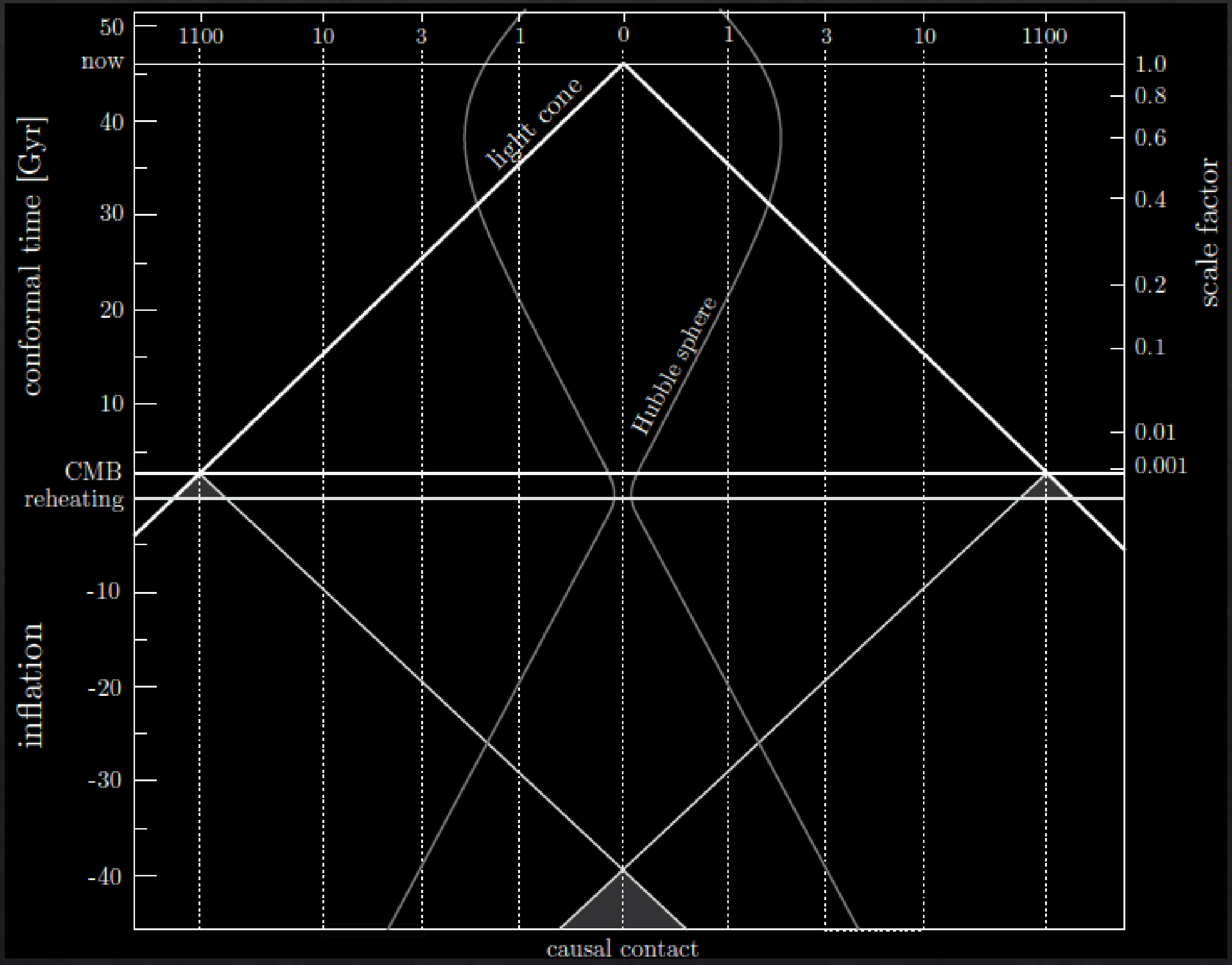
Shrinking Hubble sphere $\frac{d(aH)^{-1}}{dt} < 0$

$$\frac{d(aH)^{-1}}{dt} = H_0^{-1}\frac{1}{2}(1+3w)a^{-\frac{1}{2}(1+3w)}\dot{a} \Longrightarrow 1+3w<0$$

$$\chi_{ph} = \tau - \tau_i = \int_{t_i}^{t}\frac{dt}{a} = \int_{a_i}^{a}\frac{da}{a\dot{a}} = \int_{\ln a_i}^{\ln a}(aH)^{-1}\, d\ln a$$

$$\tau_i = \frac{2H_0^{-1}}{(1+3w)}a_i^{\frac{1}{2}(1+3w)} \xrightarrow[w>-\frac{1}{3}]{a_i \to 0} -\infty$$

Horizon Problem



Conditions for Inflation

Accelerated Expansion $\frac{d(aH)^{-1}}{dt} = \frac{d(\dot{a})^{-1}}{dt} = -\frac{\ddot{a}}{\dot{a}^2} < 0$ → $\ddot{a} > 0$

Slowly-varying Hubble parameter $\frac{d(aH)^{-1}}{dt} = \frac{\dot{a}H + a\dot{H}}{dt} = -\frac{1}{a}(1-\varepsilon) < 0$ $\varepsilon = \frac{\dot{H}}{H^2} < 1$

$$\varepsilon = -\frac{dH}{H}\frac{1}{H\,dt} = -\frac{d\ln H}{dN} < 1$$

$$\eta = -\frac{d\ln\varepsilon}{dN} = \frac{\dot{\varepsilon}}{H\varepsilon} = -\frac{d\ln H}{dN} \qquad |\eta| < 1$$

Conditions for Inflation

Quasi-de Sitter Expansion

$$\varepsilon = \frac{\dot{H}}{H^2} = 0 \Rightarrow H = \text{const}$$

$$H = \frac{\dot{a}}{a} \Leftrightarrow \dot{a} = aH \Rightarrow a = e^{Ht} \Rightarrow a^2 = e^{2Ht}$$

$$ds^2 = dt^2 - e^{2Ht}\, d\boldsymbol{x}^2$$

Scalar Field Dynamics

Inflaton $\phi(t, \boldsymbol{x}) = \bar{\phi}(t) = \phi$

Pressure $P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} < -\frac{1}{3}$$

$$V(\phi) > \frac{1}{2}\dot{\phi}^2$$

Energy density $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

Friedmann equation $H^2 = \frac{\rho_\phi}{3M_{\text{pl}}^2} = \frac{1}{3M_{\text{pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$

Klein-Gordon $\ddot{\phi} = 3H\dot{\phi} + V_{,\phi}$

Quantum fluctuations during Inflation

Inflaton perturbation $\phi(t,\boldsymbol{x}) = \bar{\phi}(t) + \delta\phi(t,\boldsymbol{x})$

Curvature perturbation $\mathcal{R} = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v)$

spatially flat gauge:
C = E = 0

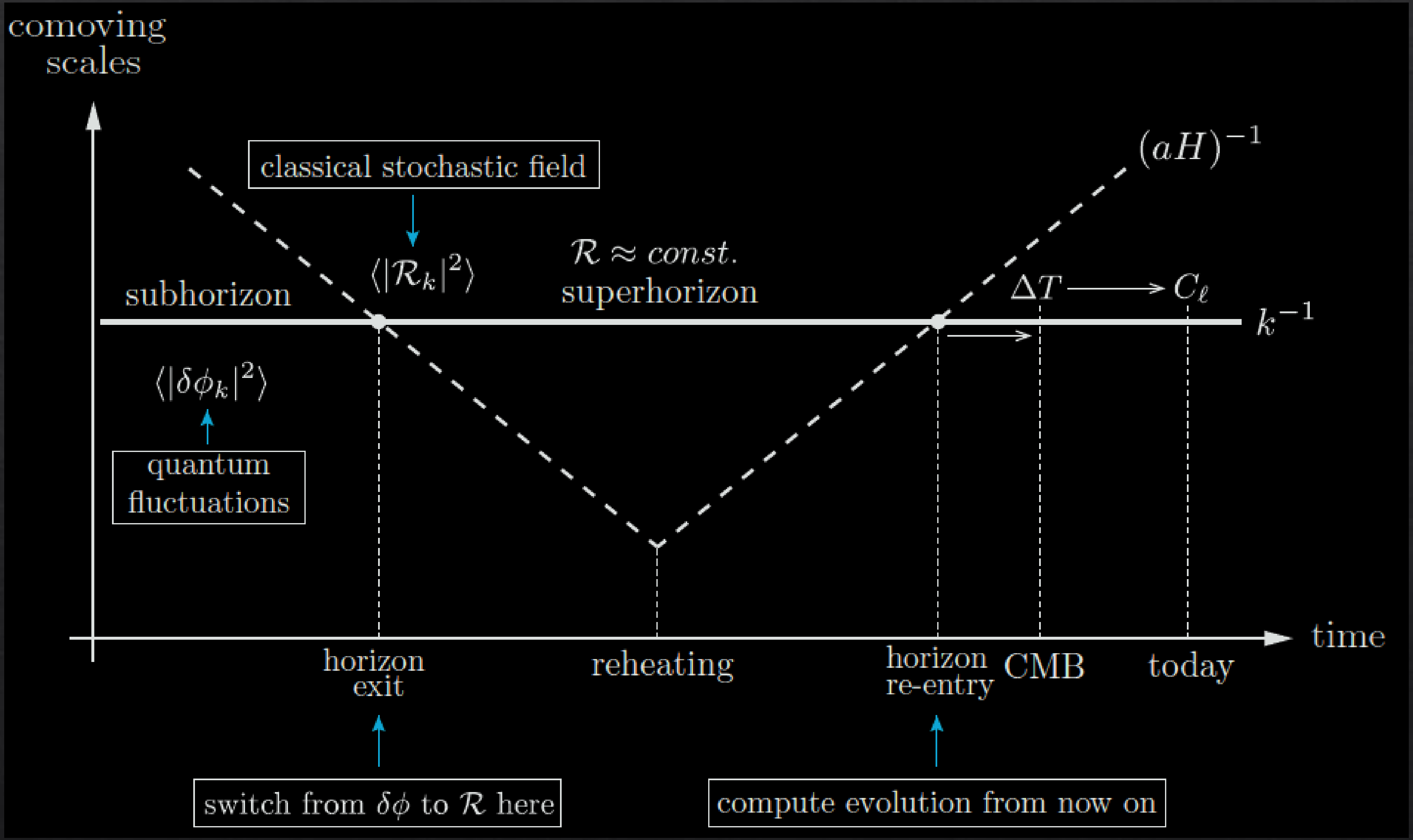
$\mathcal{R} = \mathcal{H}(B+v)$

$B + v = -\frac{\delta\phi}{\bar{\phi}'}$

$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi$

$\langle|\mathcal{R}_k|^2\rangle = \left(\frac{\mathcal{H}}{\bar{\phi}'}\right)^2 \langle|\delta\phi_k|^2\rangle$

Quantum fluctuations during Inflation



Classical Oscillators

Metric perturbation $g_{\mu\nu}(t,\boldsymbol{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\boldsymbol{x})$ $\xrightarrow[\text{C = E = 0}]{\text{spatially flat gauge:}}$ $g_{\mu\nu}(t,\boldsymbol{x}) = \bar{g}_{\mu\nu}(t)$

Flat and unperturbed FLRW metric $ds^2 = a^2(\tau)\left[d\tau^2 - \delta_{ij}dx^i dx^j\right]$

Scalar field action $S = \int d\tau d^3x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$ using

$$\phi(\tau,\boldsymbol{x}) = \bar{\phi}(t) + \overbrace{\frac{f(\tau,\boldsymbol{x})}{a(\tau)}}^{\delta\phi(t,\boldsymbol{x})}$$

$$V(\phi) = V(\bar{\phi}) + V_{,\phi}\frac{f}{a} + V_{,\phi\phi}\left(\frac{f}{a}\right)^2$$

$$S^{(2)} = \int d\tau d^3x\, a^4\left[\frac{1}{2a^2}\left(f^2\frac{a'^2}{a^4} + \frac{f'^2}{a^2} - 2f'f\frac{a'}{a^3}\right) - \frac{1}{2a^4}(\nabla f)^2 - V_{,\phi\phi}\frac{f^2}{a^2}\right] \approx \int d\tau d^3x \underbrace{\frac{1}{2}\left[(f')^2 - (\nabla f)^2 + \frac{a''}{a}f^2\right]}_{\mathcal{L}}$$

Classical Oscillators

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu f)} \right) - \frac{\partial \mathcal{L}}{\partial f} = 0 \quad \Rightarrow \quad f'' - \nabla^2 f - \frac{a''}{a} f = 0 \quad \Rightarrow \quad f''_{\boldsymbol{k}} + \left(k^2 - \frac{a''}{a} \right) f_{\boldsymbol{k}} = 0$$

$f \to f_{\boldsymbol{k}}$

$\nabla^2 f_{\boldsymbol{k}} = -k^2 f_{\boldsymbol{k}}$

Mukhanov-Sasaki Equation

Canonical Quantisation

Conjugate momentum $\pi = \dfrac{\partial \mathcal{L}}{\partial f'} = f'$

To operators
$$f(\tau, \boldsymbol{x}) \longrightarrow \hat{f}(\tau, \boldsymbol{x})$$
$$\pi(\tau, \boldsymbol{x}) \longrightarrow \hat{\pi}(\tau, \boldsymbol{x})$$
$$\Rightarrow \quad \left[\hat{f}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{x}')\right] = i\delta(\boldsymbol{x} - \boldsymbol{x}')$$

In Fourier space
$$\left[\hat{f}_{\boldsymbol{k}}(\boldsymbol{\tau}), \hat{\pi}_{\boldsymbol{k}'}(\boldsymbol{\tau})\right] = \int \frac{d^3x}{(2\pi)^{3/2}} \int \frac{d^3x'}{(2\pi)^{3/2}} \left[\hat{f}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{x}')\right] e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}'} = i\delta(\boldsymbol{k} + \boldsymbol{k}')$$

$$\hat{f}_{\boldsymbol{k}}(\tau) = f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger$$

$$\underbrace{-i\left(f_k \partial_\tau f_k^* - (\partial_\tau f_k) f_k^*\right)}_{\text{Wronskian} = 1} \times \left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger\right] = i\delta(\boldsymbol{k} + \boldsymbol{k}') \Longrightarrow \left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger\right] = i\delta(\boldsymbol{k} + \boldsymbol{k}')$$

Wronskian = 1

Vacuum State

Quasi-de Sitter Expansion $H \approx \text{const} \Rightarrow a \approx e^{Ht} \Rightarrow \frac{a''}{a} \approx \frac{2}{\tau^2}$ $\tau \longrightarrow -\infty$ $f''_{\boldsymbol{k}} + k^2 f_{\boldsymbol{k}} \approx 0$

Initial condition $\lim_{\tau \to \infty} f_{\boldsymbol{k}}(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}$

$$f_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right)$$

$\alpha = 1$
$\beta = 0$
⇒

$$f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)$$

Zero-point Fluctuations

$$\hat{f}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{f}_{\boldsymbol{k}}(\tau) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \int \frac{d^3k}{(2\pi)^{3/2}} \left[f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger\right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\langle \hat{f} \rangle \equiv \langle 0|\hat{f}(\tau, \boldsymbol{0})|0\rangle = \int \frac{d^3k}{(2\pi)^{3/2}} \langle 0|f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger|0\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0$$

$$\left\langle |\hat{f}|^2 \right\rangle \equiv \langle 0|\hat{f}^\dagger(\tau, \boldsymbol{0})\hat{f}(\tau, \boldsymbol{0})|0\rangle = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int \frac{d^3k'}{(2\pi)^{3/2}} \left\langle 0\middle|\left(f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger + f_k(\tau)\hat{a}_{\boldsymbol{k}}\right)\left(f_{k'}(\tau)\hat{a}_{\boldsymbol{k}'} + f_{k'}^*(\tau)\hat{a}_{\boldsymbol{k}'}^\dagger\right)\middle|0\right\rangle e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}'}$$

$$= \int d\ln k \, \frac{k^3}{2\pi^2} |f_k(\tau)|^2$$

Power spectrum $\quad \Delta_f^2 \equiv \frac{k^3}{2\pi^2} |f_k(\tau)|^2$

Zero-point Fluctuations

$$\delta\phi(t,\boldsymbol{x}) = \frac{f(\tau,\boldsymbol{x})}{a(\tau)} \Longrightarrow \Delta^2_{\delta\phi}(k,\tau) = \frac{\Delta^2_f(k,\tau)}{a^2(\tau)}$$

$$\Delta^2_{\delta\phi}(k,a) = \left(\frac{H}{2\pi}\right)^2 \left(1 + \left(\frac{k}{aH}\right)^2\right) \xrightarrow{k \ll aH} \Delta^2_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2$$

From Quantum to Classic

$$\hat{f}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{f}_{\boldsymbol{k}}(\tau) e^{\boldsymbol{ik \cdot x}} = \int \frac{d^3k}{(2\pi)^{3/2}} [f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger] e^{\boldsymbol{ik \cdot x}}$$

$$\hat{\boldsymbol{\pi}}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{\pi}_{\boldsymbol{k}}(\tau) e^{\boldsymbol{ik \cdot x}} = \int \frac{d^3k}{(2\pi)^{3/2}} [f'_k(\tau)\hat{a}_{\boldsymbol{k}} + (f_k^*)'(\tau)\hat{a}_{\boldsymbol{k}}^\dagger] e^{\boldsymbol{ik \cdot x}}$$

$$\mathrm{k}\tau \to 0 \qquad f_k(\tau) \approx \frac{-\mathrm{i}}{\sqrt{2}k^{3/2}\tau} \qquad f'_k(\tau) \approx \frac{\mathrm{i}}{\sqrt{2}k^{3/2}\tau^2}$$

$$\hat{f}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{f}_{\boldsymbol{k}}(\tau) e^{\boldsymbol{ik \cdot x}} = \frac{-\mathrm{i}}{\sqrt{2}\tau} \int \frac{d^3k}{(2\pi k)^{3/2}} [\hat{a}_{\boldsymbol{k}} - \hat{a}_{\boldsymbol{k}}^\dagger] e^{\boldsymbol{ik \cdot x}}$$

$$\hat{\boldsymbol{\pi}}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{\boldsymbol{\pi}}_{\boldsymbol{k}}(\tau) e^{\boldsymbol{ik \cdot x}} = \frac{\mathrm{i}}{\sqrt{2}\tau^2} \int \frac{d^3k}{(2\pi k)^{3/2}} [\hat{a}_{\boldsymbol{k}} - \hat{a}_{\boldsymbol{k}}^\dagger] e^{\boldsymbol{ik \cdot x}} = -\frac{\mathbf{1}}{\tau} \hat{f}(\tau, \boldsymbol{x})$$