# Quantum Cosmology: towards the Planck time

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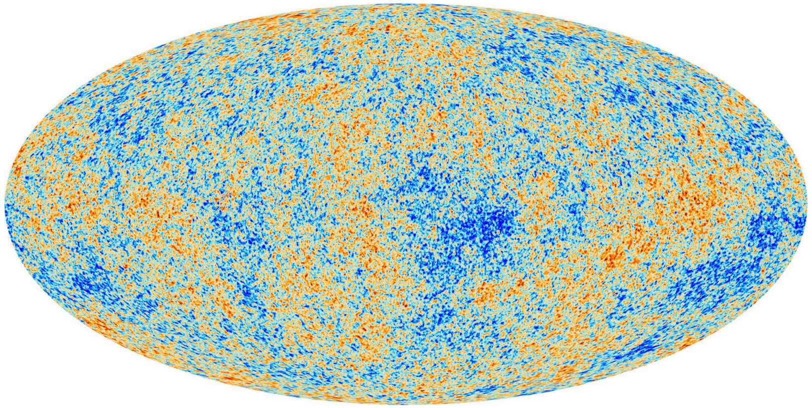
# Main reference

- Quantum Gravity and Inflation, Maria G. Romania, N. C. Tsamis, R. P. Woodard, 2012

# Introduction

- FLRW metric:
  - $ds^2 = -dt^2 + a^2(t)d\vec{x}.d\vec{x}$
- Hubble parameter:
  - $H \equiv \frac{\dot{a}}{a} = \frac{d}{dt}\ln a(t)$
- Deceleration parameter:
  - $q \equiv -\frac{\dot{a}(t)\ \ddot{a}(t)}{\dot{a}^2(t)} = -1\ -\frac{\dot{H}}{H^2(t)} = -1 + \varepsilon(t)$
- Horizon Problem



# Introduction: Single-Scalar Inflation

- Inflaton: $\varphi$

- Lagrangian:
  - $\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \; - V(\varphi) + \frac{R}{16\pi \mathrm{G}}\right)$

- This model has the condition $\dot{H}(t) \leq 0$: Weak Energy Condition.

# Introduction: Scalar Inflation Problems

- Assumes that the universe began with scalar field approximately spatially homogenous over more than a Hubble volume $V(\varphi) > H^{-3}$.
- Assuming $V(\varphi)$ must be flat enough makes the inflation last a long time.
- Assumes that the minimum of $V(\varphi)$ has just the right value $V_{min}(\varphi) \approx 0$ to leave the post-inflationary universe with only the small amount of vacuum energy we detect today.
- Assumes that the scalar field couples enough to ordinary matter so that its kinetic energy can create a hot, dense universe at the end of inflation, but not so much that loop corrections from ordinary matter compromise the flatness of $V(\varphi)$.

# Introduction: Gravity-Driven Inflation

- A more natural mechanism for inflation can be found within gravitation.
- Suppose that the bare cosmological constant $\Lambda$ is not unnaturally small but rather large and positive. “Large” means a $\Lambda$ induced by some matter scale which might be as high as $10^{18}$ GeV. Then $G\Lambda$ would be $\sim 10^{-4}$, rather than the putative of $10^{-122}$.
- $\Lambda$ is constant in space.
- $\Lambda$ is constant in time, and classical physics can’t offer a natural mechanism for stopping inflation; quantum physics can: accelerated expansion continually rips virtual infrared gravitons out of the vacuum and these gravitons attract one another, thereby slowing inflation.

# Introduction: Graviton Physical Modes

- Metric field with a fluctuating graviton field:
  - $g_{i,j} = a^2(t)\left[\delta_{i,j} + \sqrt{32\pi\mathrm{G}}\, h_{i,j}^{TT}(t,\overrightarrow{x})\right]$
- $h_{i,j}^{TT}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \{u(t,k)e^{i\,\vec{k}.\vec{x}}\varepsilon_{i,j}(\vec{k},\lambda)\alpha(\vec{k},\lambda) + (c.c.)\}$
  - $u(t,k)$: mode function;
  - $\varepsilon_{i,j}(\vec{k},\lambda)$: polarization tensor;
  - $\alpha(\vec{k},\lambda)$: annihilation operator;
  - $(c.c.)$: complex conjugate.
- $\ddot{u}(t,k) + 3H(t)\dot{u}(t,k) + \frac{k^2}{a^2(t)}u(t,k) = 0$

# … Estimate of E(t,k)

- Definition of energy is subtle for gravitons.
- Use a massless and minimally coupled scalar field $\varphi(x)$ with the same mode equation as $u(t,k)$ for the estimate.
- Lagrangian density:
  - $\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = \frac{1}{2}a^3(t)\dot{\varphi}^2 - \frac{1}{2}\nabla\varphi.\nabla\varphi$
- Lagrangian (in space):
  - $L(t) = \int d^3x\,\mathcal{L} = \int \frac{d^3k}{(2\pi)^3}\left\{\frac{1}{2}a^3(t)\left|\dot{\tilde{\varphi}}\left(t,\vec{k}\right)\right|^2 - \frac{1}{2}a(t)k^2\left|\tilde{\varphi}\left(t,\vec{k}\right)\right|^2\right\}$
- Classic harmonic oscillator Lagrangian in 1D:
  - $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2$

- Any mode with wavenumber $\vec{k}$ evolves independently as a harmonic oscillator $q(t)$.
  - $m \to a^3(t)$
  - $m\omega^2 \to a(t)k^2$ and so $\omega = \frac{k}{a}$

- $q(t) = u(t,k)A + u^*(t,k)A^\dagger, \ [A, A^\dagger] = 1$
- $E(t,k) = \frac{1}{2}a^3(t)\dot{q}^2(t) + \frac{1}{2}a(t)k^2q^2(t)$

- For de Sitter ($a(t) = e^{Ht}$)
  - $u(t,k) = \frac{H}{\sqrt{2k^3}}\left[1 - \frac{ik}{Ha(t)}\right] e^{\frac{ik}{Ha(t)}}, \quad E_{min}(t,k) = \frac{1}{2} k a^{-1}(\mathrm{t})$
    - $\langle\Omega|E(t,k)|\Omega\rangle = \frac{a^3(t)}{2}|\dot{u}(t,k)|^2 + \frac{k^2 a(t)}{2}|u(t,k)|^2 = \frac{k}{a(t)}\left(\frac{1}{2} + \left[\frac{Ha(t)}{2k}\right]^2\right)$
      - $|\Omega\rangle$: Bunch-Davies vacuum- minimum energy state in a distant past.
      - Quantum harmonic oscillator: $\left(\frac{1}{2} + N\right)\hbar\omega$ so $N = \left[\frac{Ha(t)}{2k}\right]^2$.

- $k_{phys} = \frac{k}{a(t)}$, Horizon crossing happens when $k_{phys} = \frac{k}{a(t)} = H$
  - Infrared modes: $H < k < Ha(t)$
  - Ultraviolet modes: $k > Ha(t)$

# Model Building: Perturbative Results

- Here we consider de Sitter inflation: $a(t) = e^{Ht}$

- Gravitational Lagrangian is:
  - $\mathcal{L}_{gr} = \frac{\sqrt{-g}}{16\pi G}(R - 2\Lambda)$
- Quantum corrections should not grow faster than powers of $\ln(a) = Ht$.
- In the regime $\ln(a) \gg 1$, the energy density induced by quantum gravitational effects are:
  - $\rho_1 \sim \Lambda^2$
  - $\rho_2 \sim -G\Lambda^3 \ln[a(t)]$
  - $\rho_L \sim -\Lambda^2(G\Lambda \ln[a(t)])^{L-1}$, $L$ is the order of the loop.

- Stress-energy is separately conserved at each loop order:
  - $\dot{\rho_L} = -3H(\rho_L + p_L) \Rightarrow p_L(t) \sim -\rho_L(t)$

- General form of $p$ is:
  - $p(t) \sim \Lambda^2 f[G\Lambda \ln(a)]$

- If $G\Lambda \ln(a)$ is small, petrubation theory is valid. It breaks down when the number of e-foldings $N \equiv Ht = \ln(a) \sim (G\Lambda)^{-1}$

# Guessing the Effective Field Equations

- With $\mathcal{L}_{gr} = \frac{\sqrt{-g}}{16\pi G}(R - 2\Lambda)$ the classical gravitational equations of motion are:
  - $G_{\mu\nu} = -\Lambda g_{\mu\nu}$

- … adding a quantum induced stress-energy tensor $T_{\mu\nu}[g]$:
  - $G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}[g]$

- Principles $T_{\mu\nu}[g]$ must obey:
  - Correspondence
  - Non-locality
  - Causality

# Perfect Fluid Ansatz

- Considering $T_{\mu\nu}[g]$ is in the form of a perfect fluid:
  - Can represent any cosmology.
  - Provides enough free parameters to enforce conservation.

- $T_{\mu\nu}[g] = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$
  - $\rho, p$ and $u_\mu$ are functionals of the metric tensor.
  - $u_\mu$ is chosen to be timelike and normalized:
    - $g^{\mu\nu} u_\mu u_\nu = -1 \quad \Rightarrow \quad u^\mu u_{\mu;\nu} = 0$

- Hence there are five independent functionals.

- $\nabla^\mu T_{\mu\nu} = 0$ gives 4 equations and allows to determine any 4 of the 5 functionals in terms of the fifth.

- Specify $p[g]$.

# p[g]

- To be a casual, non-local functional of the metric, one may use the following ansatz:
  - $p[g](x) = \Lambda^2 f\big(-G\Lambda X[g](x)\big) \xrightarrow{de\ Sitter} \Lambda^2 f[G\Lambda \ln(a)]$
  - $-X[g](x)$: dimensionless and non-local functional of the metric.

- The inverse of some differential operator introduces causal non-locality
  - Covariant scalar d'Alembertian: $\Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu\big(g^{\mu\nu}\sqrt{-g}\partial_\nu\big)$

- Acting the inverse of $\Box$ on a curvature scalar ($R$ for example) gives a dimensionless result.

- $X[g] = \Box^{-1}R$

- In FLRW geometry:
  - $\Box = -(\partial_t^2 + 3H\partial_t)$
  - $R = 12H^2(t) + 6\,\dot{H}(t)$
- Then:
  - $X[g] = \Box^{-1}R = -\int_0^t dt'\, a^{-3} \int_0^{t'} dt'' a^3 \left[12H^2 + 6\,\dot{H}\right] \xrightarrow{de\ Sitter} = -4\ln(a) + \frac{4}{3}[1 - e^{-3Ht}].$
- This means that $-X[g]$ grows during the inflationary regime of large $R$, and is approximately constant during the period of radiation domination ($R(t) = 0$ because $a(t) \propto t^{\frac{1}{2}}$ at this period).
- $f\left(-G\Lambda X[g](x)\right)$ must be a function monotonically crescent and without bound for the pressure to produce enough screening to end inflation in roughly the right way.

# Numerical Results

- Using the exponential model ($f(x) = e^x - 1$) and $G\Lambda = \frac{1}{200}$, the authors made numerical works and obtained graphs for $X(t), R(t), H(t), \dot{H}(t), a(t)$ and $q(t)$.

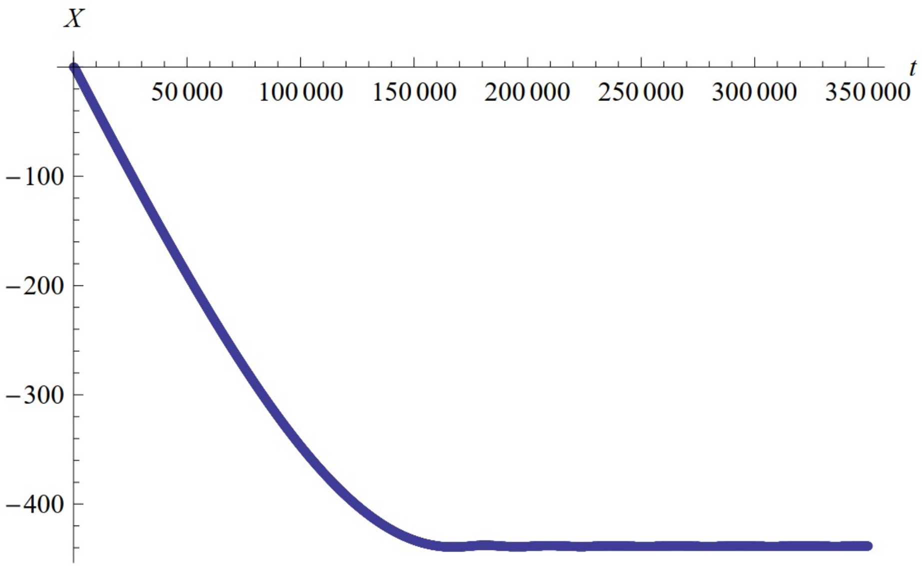



**Figure** 1: The evolution of the source $X(t)$ over the full range for the exponential model.

Inflation dominates until near $X_{cr}$, defined as: $1 - 8\pi G\Lambda f[-G\Lambda X_{cr}] = 0$

$X(t)$ oscillations have constant period and decreasing amplitude

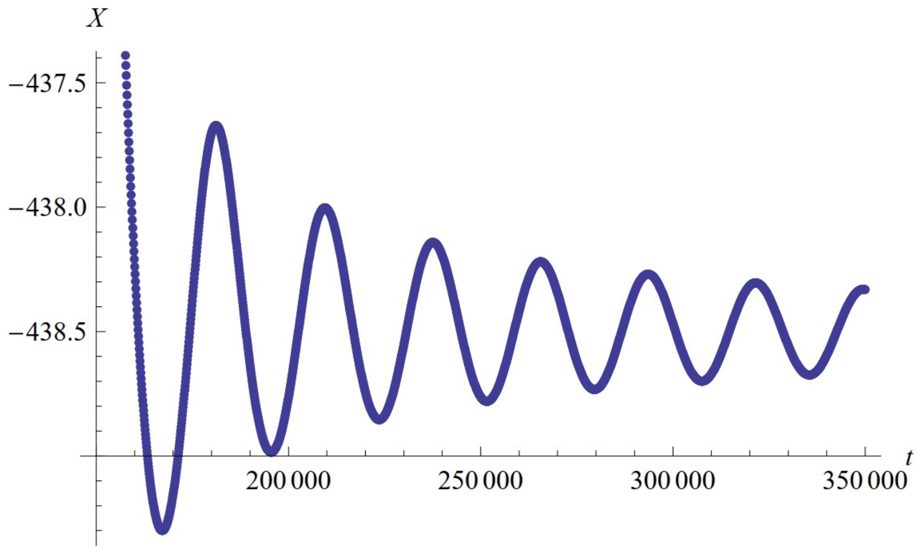



**Figure** 2: The evolution of the source $X(t)$ during the oscillatory regime for the exponential model.

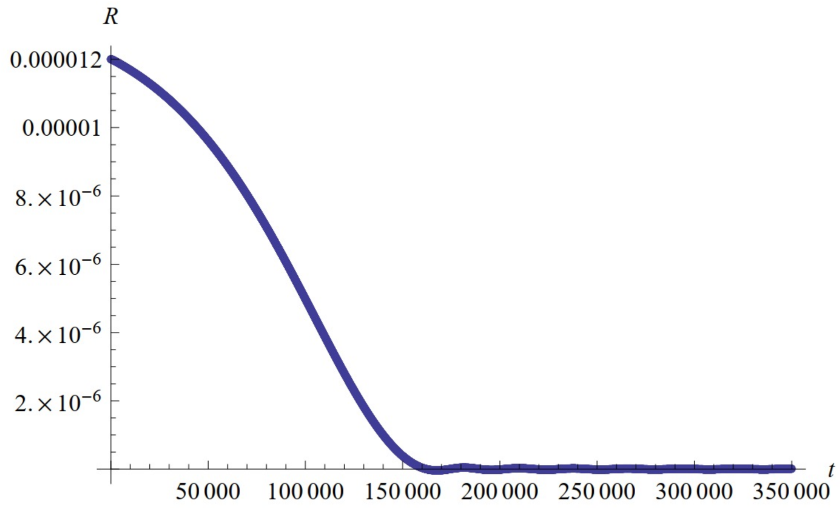



**Figure 3:** The evolution of the curvature scalar $R(t)$ over the full range for the exponential model.

$R(t)$ oscillations are centered at $R = 0$, the amplitude grows like $t^{-1}$ and have $\omega = G\Lambda H_0\sqrt{72\pi f'_{cr}}$.

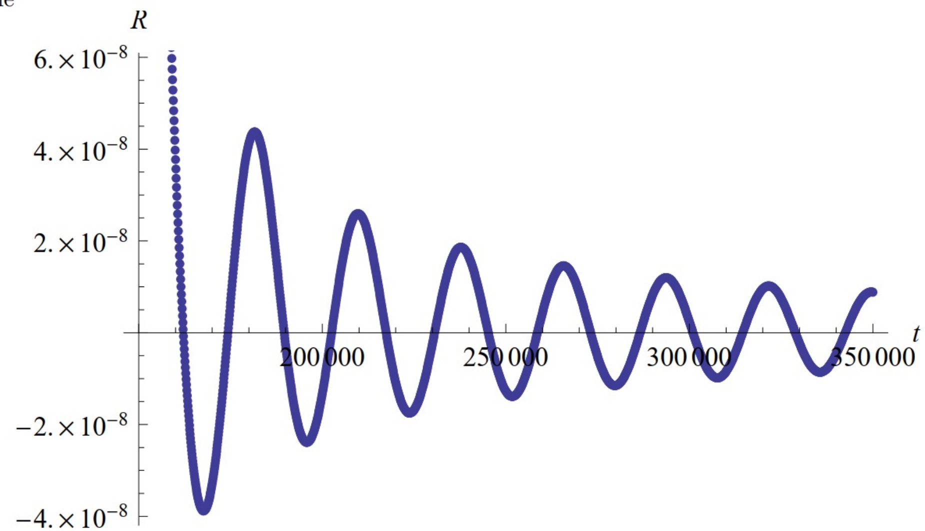



**Figure 4:** The evolution of the curvature scalar $R(t)$ during the oscillatory regime for the exponential model.

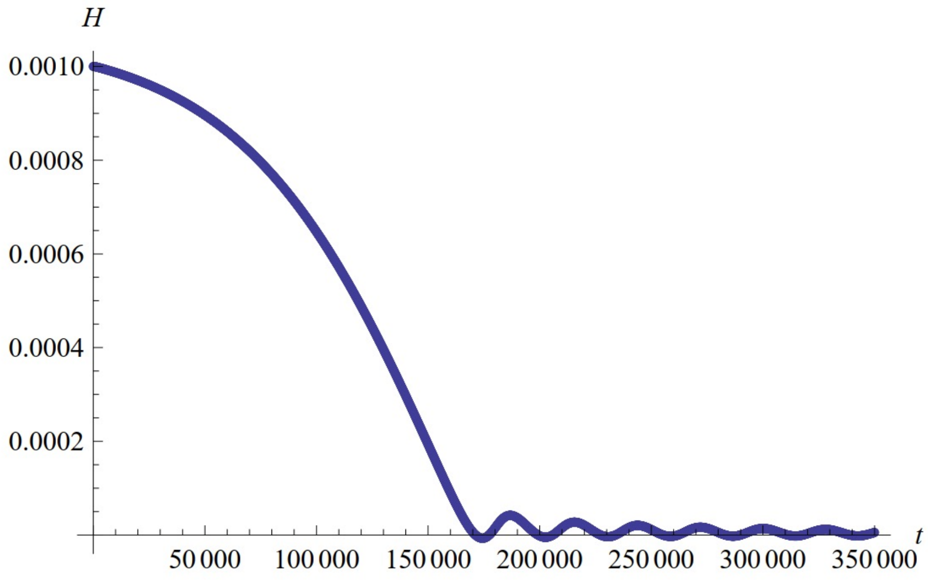



**Figure** 5: The evolution of the Hubble parameter $H(t)$ over the full range for the exponential model.

There is net expansion, however $H(t)<0$ in some small time intervals where the universe is compressing. This may explain rapid reheating when matter couplings are included.

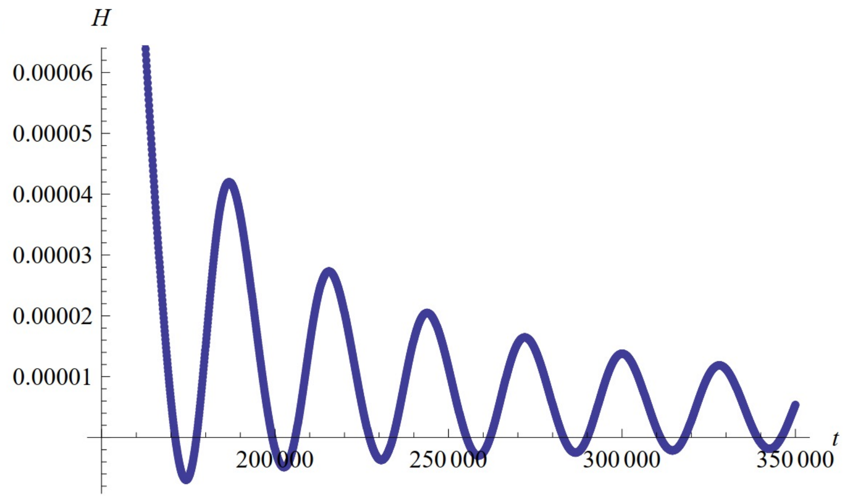



**Figure** 6: The evolution of the Hubble parameter $H(t)$ during the oscillatory regime for the exponential model.

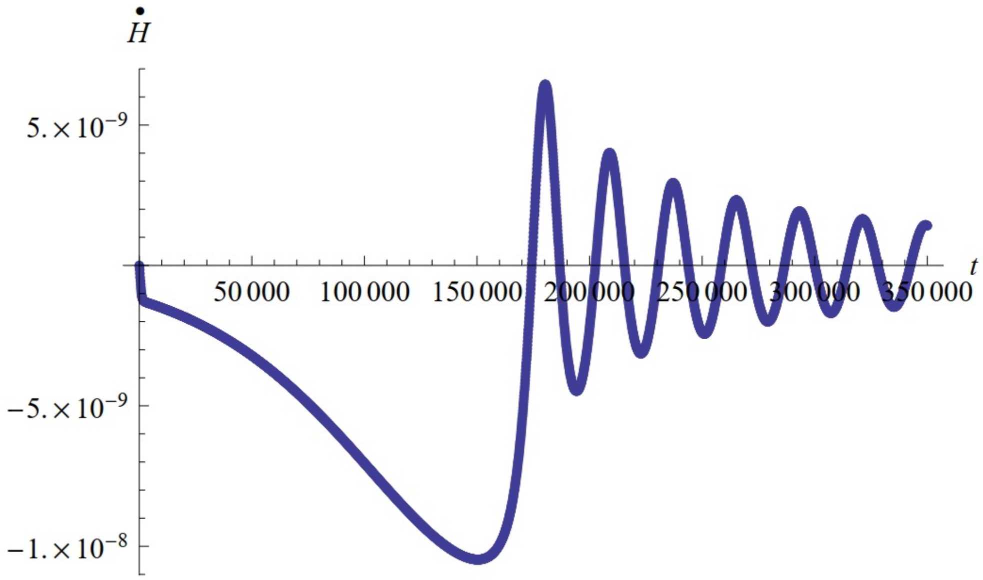



**Figure** 7: The evolution of $\dot{H}(t)$ over the full range for the exponential model.

During oscillations $\dot{H} > 0$ for about half of the time which violates the weak energy condition ($\dot{H} \leq 0, q \geq -1$). This the hallmark of a quantum effect.

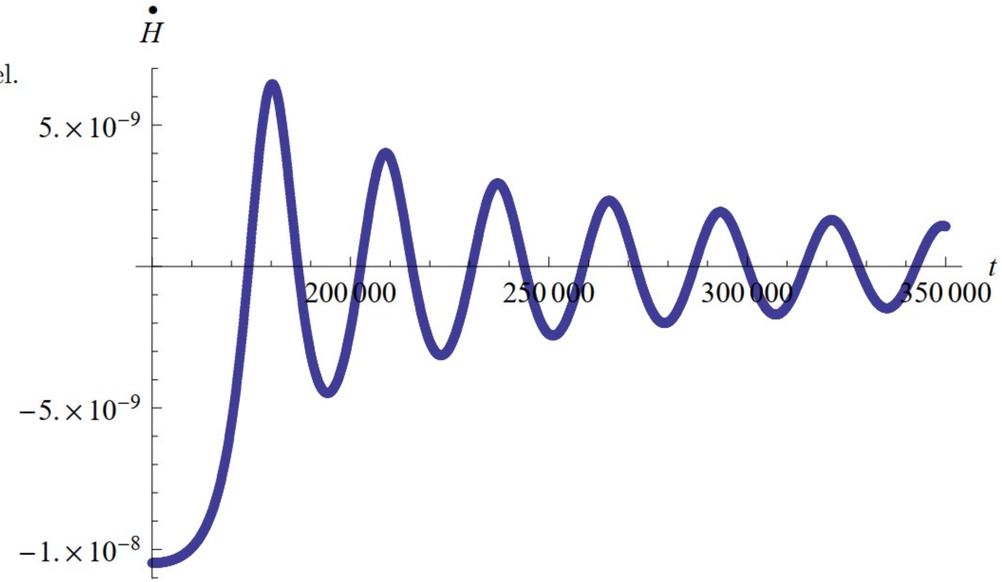



**Figure** 8: The evolution of $\dot{H}(t)$ during the oscillatory regime for the exponential model.

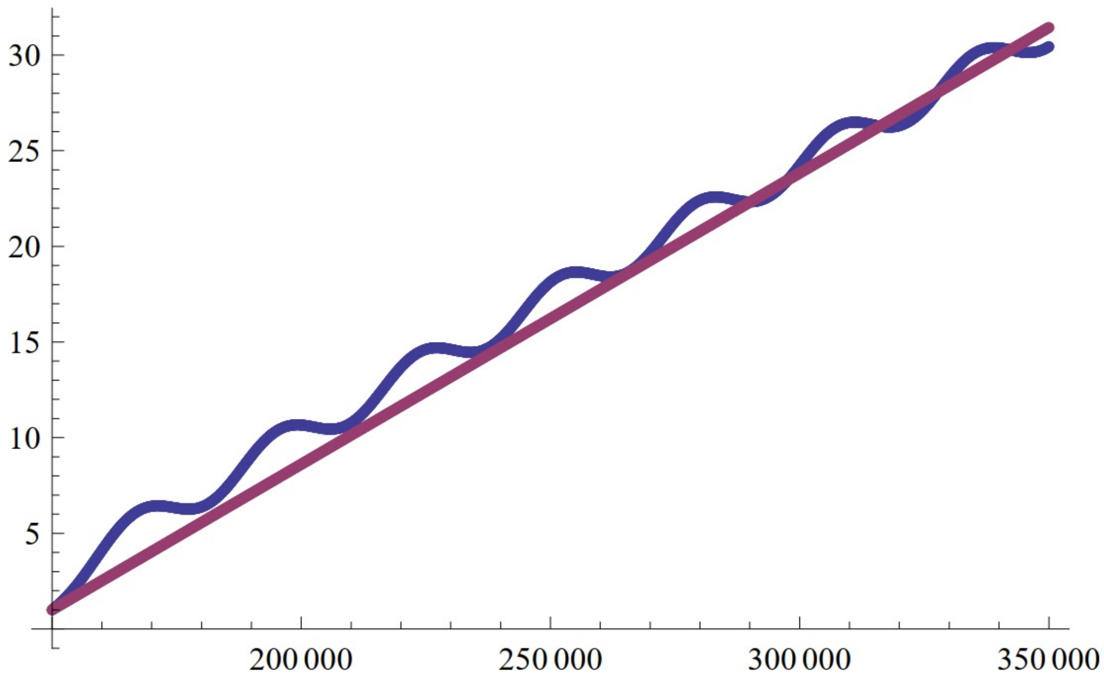



**Figure 11:** The evolution of the scale factor ratio $[a(t)/a(150000)]$ during the oscillatory regime for the exponential model *versus* a linear interpolation.

$a(t)$ oscillations are centered around a linear increase with time, which may be a problem for reheating.

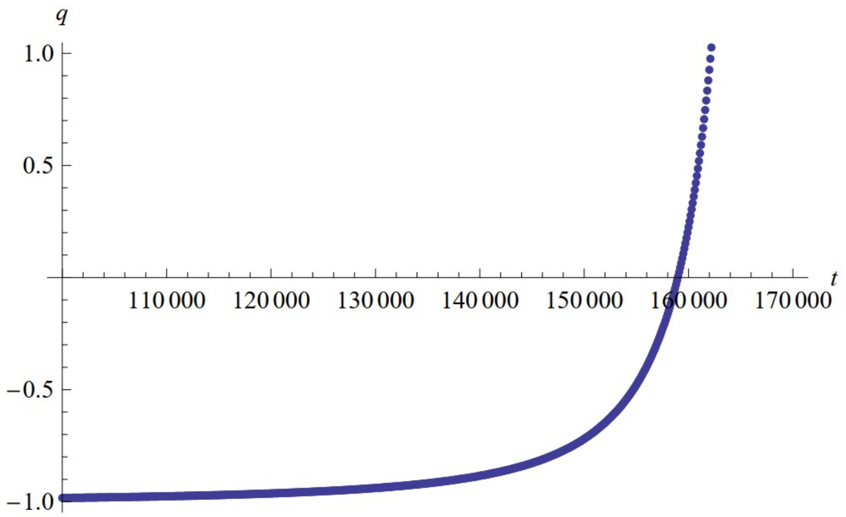



**Figure** 10: The evolution of the decelaration parameter $q(t)$ around the end of inflation for the exponential model.

The epoch of inflation ($q < 0$) ends slightly before $X(t)$ reaches $X_{cr}$.

$q(t = 0) = -1$ and $q(t = t_{cr}) = \frac{1}{2}$.

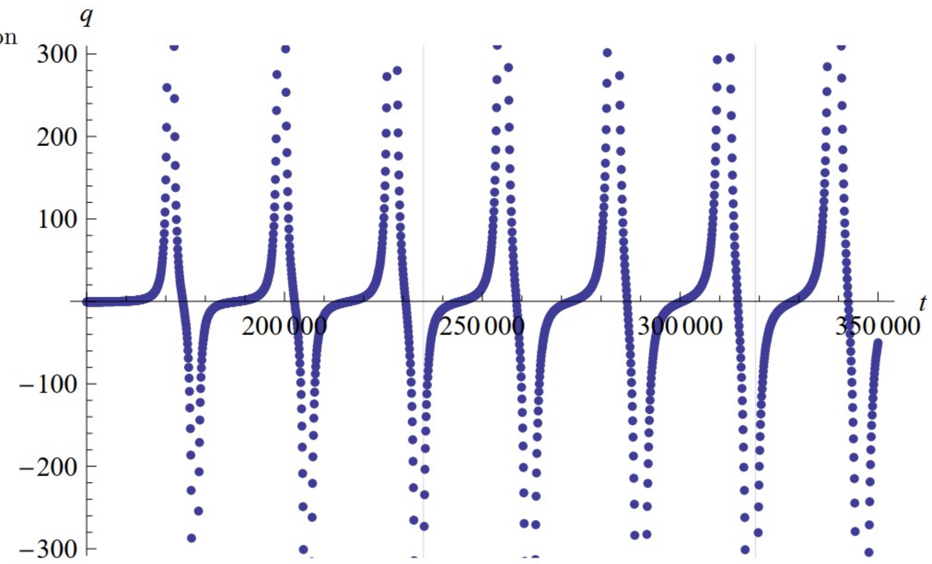



**Figure** 9: The evolution of the deceleration parameter $q(t)$ during the oscillatory regime for the exponential model.

During oscillations q $< -1$ for some time intervals which violates the weak energy condition ($\dot{H} \leq 0, q \geq -1$).

# Analytical Results

- $2\dot{H} + 3H^2 = \Lambda - 8\pi G p = \Lambda\{1 - 8\pi G\Lambda f[-G\Lambda X]\}$
- $f(x)$ is growing monotonically and without bound so there is $X_{cr}$ such that $1 - 8\pi G\Lambda f[-G\Lambda X_{cr}] = 0$
- Inflationary evolution dominates roughly until the $X_{cr}$. Near $X_{cr}$ the quantum induced pressure $p$ is nearly constant:
  - $f \approx f_{cr} - G\Lambda\Delta X(t) f'_{cr},\ \Delta X(t) \equiv X(t) - X_{cr}.$
- $R(t) \approx \frac{\sin(\omega t)}{a(t)}$. $\omega$ is the same as the calculated one in the numerical work.

# Post Inflationary Evolution

- Energy flows from the gravitational to the matter sector, leading to the radiation domination period at $t_r$. Here $R = 0$ and the quantum induced stress-energy simply cancels the bare cosmological constant.

- At $t = t_m$ it's possible to use first order perturbation theory to compute the total pressure ($p_m = 0$):
  - $p_{tot} \equiv -\frac{\Lambda}{8\pi G} + p[g](x) \approx -\frac{\Lambda}{G}(G\Lambda)^2 f'_{cr}\Delta X, \quad \Delta X \equiv X(t) - X_{cr} = -\frac{4}{3}\ln\left[1 + \frac{3}{2}H_m(t - t_m)\right] + O(1)$

- However, at matter domination period there are two problems:
  - $f' > 0$ and $f$ is unbounded so $p_{tot} > 0$ when $X(t) < X_{cr} \ll 0$. The observation of late time acceleration implies negative pressure.
  - $\left(\frac{p_{tot}}{p_{now}}\right) \approx \left(\frac{G\Lambda \mathrm{H}_I}{H_{now}}\right)^2 f'_{cr}\Delta X \approx 10^{86} f'_{cr}\Delta X, \quad p_{now} \approx -\frac{3}{8\pi \mathrm{G}} H_{now}^2, \ H_I \sim 10^{13} GeV,$ $H_{now} \sim 10^{-33} eV$. The magnitude of $p_{tot}$ is too large.

# References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