Kinetic Theory

The gaseous condition is exemplified in the soirée, where the members rush about confusedly, and the only communication is during a collision, which in some instances may be prolonged by button-holing.

JAMES CLERK MAXWELL (1873)

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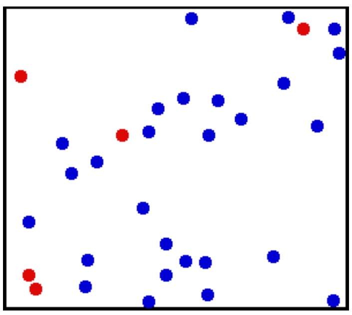
Kinetic Theory



Kinetic theory deals with the statistical distribution of a "gas" made from a huge number of "particles" that travel freely, without collisions, for distances (mean free paths) long compared to their sizes.

In kinetic theory, the key concept is the distribution function, or number density of particles in phase space, $\mathcal{N}$, that is, the number of particles of some species (e.g., electrons) per unit of physical space and of momentum space.

This $\mathcal{N}$ and the frame-independent laws it obeys provide us with a means for computing, from microphysics, the macroscopic quantities of continuum physics: mass density, thermal energy density, pressure, equations of state, thermal and electrical conductivities, viscosities, diffusion coefficients, ...

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Examples:

- Whether neutrons in a nuclear reactor can survive long enough to maintain a nuclear chain reaction and keep the reactor hot.
- How galaxies, formed in the early universe, congregate into clusters as the universe expands.
- How spiral structure develops in the distribution of a galaxy's stars.
- How, deep inside a white-dwarf star, relativistic degeneracy influences the equation of state of the star's electrons and protons.
- How a supernova explosion affects the evolution of the density and temperature of interstellar molecules.
- How anisotropies in the expansion of the universe affect the temperature distribution of the cosmic microwave photons—the remnants of the big bang.
- How changes of a metal's temperature affect its thermal and electrical conductivity (with the heat and current carried by electrons).

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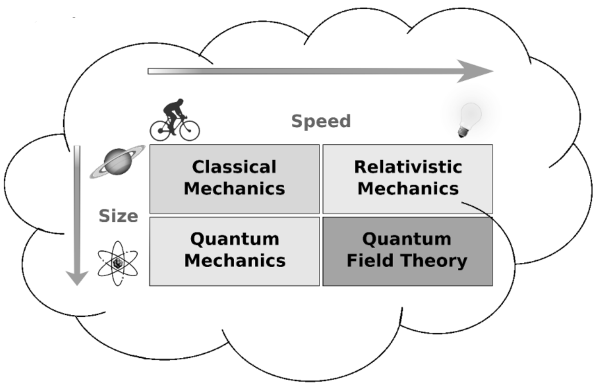
KT in different limits



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Real (physical) and momentum spaces

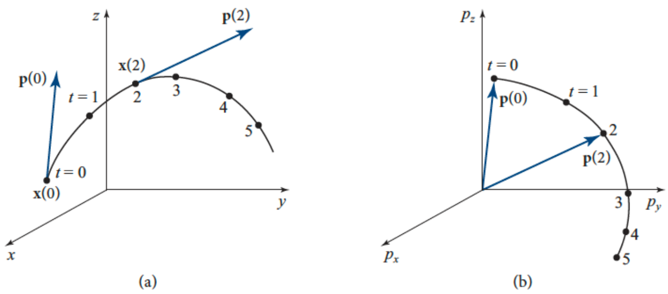

FIGURE 3.1 (a) Euclidean physical space, in which a particle moves along a curve $\mathbf{x}(t)$ that is parameterized by universal time t. In this space, the particle's momentum $\mathbf{p}(t)$ is a vector tangent to the curve. (b) Momentum space, in which the particle's momentum vector $\mathbf{p}$ is placed, unchanged, with its tail at the origin. As time passes, the momentum's tip sweeps out the indicated curve $\mathbf{p}(t)$.

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Phase space

The 3-dimensional physical space and 3-dimensional momentum space together constitute a 6-dimensional phase space, with coordinates {x, y, z, p_x , p_y , p_z}.

Consider the 6-dimensional volume $d^2V \equiv dV_x dV_p$.

In any Cartesian coordinate system, we can think of dV_x as a cube located at (x, y, z) with edge lengths dx, dy, dz, and similarly for dV_p. Then, as computed in this coordinate system, these volumes are

dV_x= dx dy dz, dV_p= dp_x dp_y dp_z,

and

d^2V = dx dy dz dp_x dp_y dp_z.

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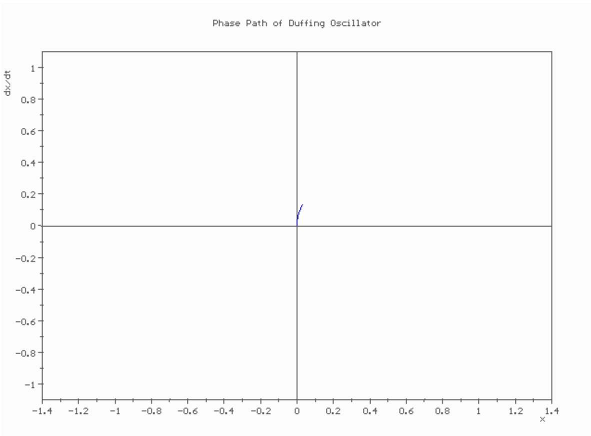
Phase space

Direct product of direct space and reciprocal space



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Newtonian distribution function

The number density of particles at location (x, p) in phase space at time t

$$\mathcal{N}(\mathbf{x}, \mathbf{p}, t) \equiv \frac{dN}{d^2\mathcal{V}} = \frac{dN}{d\mathcal{V}_x\, d\mathcal{V}_p}$$

is called the particle distribution function.

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Newtonian and relativistic distribution function $\mathcal{N}$

In Newtonian theory, the volumes dV_x and dV_p occupied by our collection of dN particles are independent of the reference frame that we use to view them.

Not so in relativity theory: dV_x undergoes a Lorentz contraction when one views it from a moving frame, and dV_p also changes; but (as we shall see) their product $d^2V = dV_x dV_p$ is the same in all frames.

Therefore, in both Newtonian theory and relativity theory, the distribution function $\mathcal{N} = dN/d^2V$ is independent of reference frame, and also, of course, independent of any choice of coordinates.

$\mathcal{N}$ is a coordinate independent scalar in phase space.

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Relativistic distribution function

Spacetime

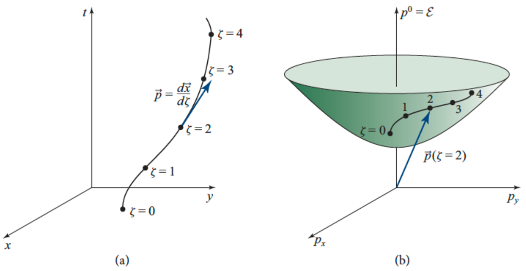

FIGURE 3.2 (a) The world line $\vec{x}(\zeta)$ of a particle in spacetime (with one spatial coordinate, z, suppressed), parameterized by a parameter ζ that is related to the particle's 4-momentum by $\vec{p} = d\vec{x}/d\zeta$. (b) The trajectory of the particle in momentum space. The particle's 4-momentum is confined to the mass hyperboloid, $\vec{p}^2 = -m^2$ (also known as the mass shell).

$$\vec{u} \equiv d\mathcal{P}/d\tau = d\vec{x}/d\tau$$

$$\vec{p} = m\vec{u}, \qquad \zeta = \tau/m$$

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Momentum space and mass hyperboloid

The momentum-space diagram drawn in Fig. 3.2b has as its coordinate axes the components (p^0, $p^1 = p_1 \equiv p_x$, $p^2 = p_2 \equiv p_y$, $p^3 = p_3 \equiv p_z$) of the 4-momentum as measured in some arbitrary inertial frame. Because the squared length of the 4-momentum is always $-m^2$,

$$\vec{p} \cdot \vec{p} = -(p^0)^2 + (p_x)^2 + (p_y)^2 + (p_z)^2 = -m^2, \tag{3.4c}$$

the particle's 4-momentum (the tip of the 4-vector $\vec{p}$) is confined to a hyperboloid in momentum space. This *mass hyperboloid* requires no coordinates for its existence; it is the frame-independent set of points in momentum space for which $\vec{p} \cdot \vec{p} = -m^2$.

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$$\mathcal{E} \equiv p^0 \tag{3.4d}$$

(with the $\mathcal{E}$ in script font to distinguish it from the energy $E = \mathcal{E} - m$ with rest mass removed and its nonrelativistic limit $E = \frac{1}{2}mv^2$), and we embody the particle's spatial momentum in the 3-vector $\mathbf{p} = p_x\mathbf{e}_x + p_y\mathbf{e}_y + p_z\mathbf{e}_z$. Therefore, we rewrite the mass-hyperboloid relation (3.4c) as

$$\mathcal{E}^2 = m^2 + |\mathbf{p}|^2. \tag{3.4e}$$

If no forces act on the particle, then its momentum is conserved, and its location in momentum space remains fixed. A force (e.g., due to an electromagnetic field) pushes the particle's 4-momentum along some curve in momentum space that lies on the mass hyperboloid. If we parameterize that curve by the same parameter ζ as we use in spacetime, then the particle's trajectory in momentum space can be written abstractly as $\vec{p}(\zeta)$. Such a trajectory is shown in Fig. 3.2b.

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Phase space

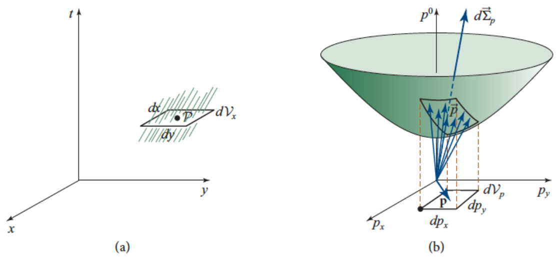

FIGURE 3.3 Definition of the distribution function from the viewpoint of a specific observer in a specific inertial reference frame, whose coordinate axes are used in these drawings. (a) At the event $\mathcal{P}$, the observer selects a 3-volume $d\mathcal{V}_x$ and focuses on the set $\mathcal{S}$ of particles that lie in $d\mathcal{V}_x$. (b) These particles have momenta lying in a region of the mass hyperboloid that is centered on $\vec{p}$ and has 3-momentum volume $d\mathcal{V}_p$. If dN is the number of particles in that set $\mathcal{S}$, then $\mathcal{N}(\mathcal{P}, \vec{p}) \equiv dN/d\mathcal{V}_x d\mathcal{V}_p$.

This 7- or 8-dimensional phase space, by contrast with the nonrelativistic 6-dimensional phase space, is frame independent. No coordinates or reference frame are actually needed to define spacetime and explore its properties, and none are needed to define and explore 4-momentum space or the mass hyperboloid—though inertial (Lorentz) coordinates are often helpful in practical situations.

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Volumes in phase space and distribution function

Now turn attention from an individual particle to a collection of a huge number of identical particles, each with the same rest mass m, and allow m to be finite or zero (it does not matter which). Examine those particles that pass close to a specific event $\mathcal{P}$ (also denoted $\vec{x}$) in spacetime; and *examine them from the viewpoint of a specific observer, who lives in a specific inertial reference frame.* Figure 3.3a is a spacetime diagram drawn in that observer's frame. As seen in that frame, the event $\mathcal{P}$ occurs at time t and spatial location (x, y, z).

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Specifically, the observer, in her inertial frame, chooses a tiny 3-volume

$$d\mathcal{V}_x = dx\, dy\, dz \tag{3.5a}$$

centered on location $\mathcal{P}$ (little horizontal rectangle shown in Fig. 3.3a) and a tiny 3-volume

$$d\mathcal{V}_p = dp_x\, dp_y\, dp_z \tag{3.5b}$$

centered on $\mathbf{p}$ in momentum space (little rectangle in the p_x-p_y plane in Fig. 3.3b). Ask the observer to focus on the set $\mathcal{S}$ of particles that lie in $d\mathcal{V}_x$ and have spatial momenta in $d\mathcal{V}_p$ (Fig. 3.3). If there are dN particles in this set $\mathcal{S}$, then the observer will identify

$$\mathcal{N} \equiv \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} \equiv \frac{dN}{d^2\mathcal{V}} \tag{3.6}$$

as the number density of particles in phase space or *distribution function*.

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PROOF OF FRAME INDEPENDENCE OF $\mathcal{N} = dN/d^2\mathcal{V}$

To prove the frame independence of $\mathcal{N}$, we shall consider the frame dependence of the spatial 3-volume $d\mathcal{V}_x$, then the frame dependence of the momentum 3-volume $d\mathcal{V}_p$, and finally the frame dependence of their product $d^2\mathcal{V} = d\mathcal{V}_x d\mathcal{V}_p$ and thence of the distribution function $\mathcal{N} = dN/d^2\mathcal{V}$.

The thing that identifies the 3-volume $d\mathcal{V}_x$ and 3-momentum $d\mathcal{V}_p$ is the set of particles $\mathcal{S}$. We select that set once and for all and hold it fixed, and correspondingly, the number of particles dN in the set is fixed. Moreover, we assume that the particles' rest mass m is nonzero and shall deal with the zero-rest-mass case at the end by taking the limit $m \to 0$. Then there is a preferred frame in which to observe the particles $\mathcal{S}$: their own rest frame, which we identify by a prime.

In their rest frame and at a chosen event $\mathcal{P}$, the particles $\mathcal{S}$ occupy the interior of some box with imaginary walls that has some 3-volume $d\mathcal{V}_{x'}$. As seen in some other "laboratory" frame, their box has a Lorentz-contracted volume $d\mathcal{V}_x = \sqrt{1-v^2}\, d\mathcal{V}_{x'}$. Here v is their speed as seen in the laboratory frame. The Lorentz-contraction factor is related to the particles' energy, as measured in the laboratory frame, by $\sqrt{1-v^2} = m/\mathcal{E}$, and therefore $\mathcal{E} d\mathcal{V}_x = m d\mathcal{V}_{x'}$. The right-hand side is a frame-independent constant m times a well-defined number that everyone can agree on: the particles' rest-frame volume $d\mathcal{V}_{x'}$, i.e.,

$$\boxed{\mathcal{E} d\mathcal{V}_x = (\text{a frame-independent quantity}).} \tag{3.7a}$$

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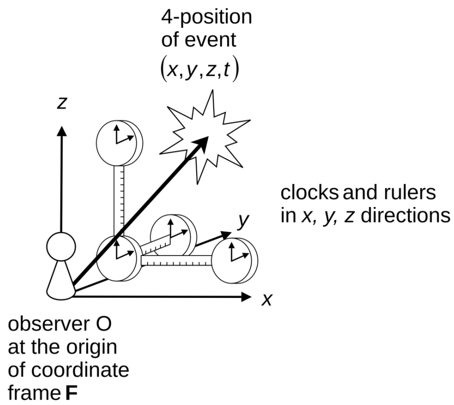
Thus, the spatial volume $d\mathcal{V}_x$ occupied by the particles is frame dependent, and their energy $\mathcal{E}$ is frame dependent, but the product of the two is independent of reference frame.



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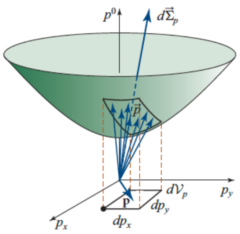
Turn now to the frame dependence of the particles' 3-volume $d\mathcal{V}_p$. As one sees from Fig. 3.3b, $d\mathcal{V}_p$ is the projection of the frame-independent mass-hyperboloid region $d\vec{\Sigma}_p$ onto the laboratory's xyz 3-space. Equivalently, it is the time component $d\Sigma_p^0$ of $d\vec{\Sigma}_p$. Now, the 4-vector $d\vec{\Sigma}_p$, like the 4-momentum $\vec{p}$, is orthogonal to the mass hyperboloid at the common point where they intersect it, and therefore $d\vec{\Sigma}_p$ is parallel to $\vec{p}$. This means that, when one goes from one reference frame to another, the time components of these two vectors will grow or shrink in the same manner: $d\Sigma_p^0 = d\mathcal{V}_p$ is proportional to $p^0 = \mathcal{E}$, so their ratio must be frame independent:



$$\boxed{\frac{d\mathcal{V}_p}{\mathcal{E}} = \text{(a frame-independent quantity)}.} \quad (3.7b)$$

(If this sophisticated argument seems too slippery to you, then you can develop an alternative, more elementary proof using simpler 2-dimensional spacetime diagrams: Ex. 3.1.)

By taking the product of Eqs. (3.7a) and (3.7b) we see that for our chosen set of particles $\mathcal{S}$,

$$d\mathcal{V}_x d\mathcal{V}_p = d^2\mathcal{V} = \text{(a frame-independent quantity)}; \quad (3.7c)$$

and since the number of particles in the set, dN, is obviously frame-independent, we conclude that

$$\boxed{\mathcal{N} = \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} \equiv \frac{dN}{d^2\mathcal{V}} = \text{(a frame-independent quantity)}.} \quad (3.8)$$

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Exercise 3.1 *Derivation and Practice: Frame Dependences of $d\mathcal{V}_x$ and $d\mathcal{V}_p$* R T2

Use the 2-dimensional spacetime diagrams of Fig. 3.4 to show that $\mathcal{E} d\mathcal{V}_x$ and $d\mathcal{V}_p/\mathcal{E}$ are frame independent [Eqs. (3.7a) and (3.7b)].

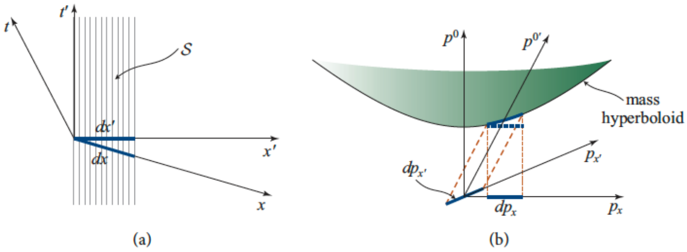

FIGURE 3.4 (a) Spacetime diagram drawn from the viewpoint of the (primed) rest frame of the particles $\mathcal{S}$ for the special case where the laboratory frame moves in the $-x'$ direction with respect to them. (b) Momentum-space diagram drawn from viewpoint of the unprimed observer.

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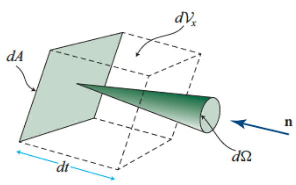
Distribution function for photons



When dealing with photons or other zero-rest-mass particles, one often expresses $\mathcal{N}$ in terms of the *specific intensity* I_ν. This quantity is defined as follows (see Fig. 3.5). An observer places a CCD (or other measuring device) perpendicular to the photons' propagation direction **n**—perpendicular as measured in her reference frame. The region of the CCD that the photons hit has surface area dA as measured by her, and because the photons move at the speed of light c, the product of that surface area with c times the time dt that they take to all go through the CCD is equal to the volume they occupy at a specific moment of time:

$$d\mathcal{V}_x = dA\, c dt. \qquad (3.11a)$$

Focus attention on a set $\mathcal{S}$ of photons in this volume that all have nearly the same frequency ν and propagation direction **n** as measured by the observer. Their energies $\mathcal{E}$ and momenta **p** are related to ν and **n** by

$$\mathcal{E} = h\nu, \quad \mathbf{p} = (h\nu/c)\mathbf{n}, \qquad (3.11b)$$

where h is Planck's constant. Their frequencies lie in a range $d\nu$ centered on ν, and they come from a small solid angle $d\Omega$ centered on $-\mathbf{n}$; the volume they occupy in momentum space is related to these quantities by

$$d\mathcal{V}_p = |\mathbf{p}|^2 d\Omega d|\mathbf{p}| = (h\nu/c)^2 d\Omega (h d\nu/c) = (h/c)^3 \nu^2 d\Omega d\nu. \qquad (3.11c)$$

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The photons' specific intensity, as measured by the observer, is defined to be the total energy

$$d\mathcal{E} = h\nu dN \quad (3.11d)$$

(where dN is the number of photons) that crosses the CCD per unit area dA, per unit time dt, per unit frequency $d\nu$, and per unit solid angle $d\Omega$ (i.e., per unit everything):

$$I_\nu \equiv \frac{d\mathcal{E}}{dAdtd\nu d\Omega}. \quad (3.12)$$

(This I_ν is sometimes denoted $I_{\nu\Omega}$.) From Eqs. (3.8), (3.11), and (3.12) we readily deduce the following relationship between this specific intensity and the distribution function:

$$\boxed{\mathcal{N} = \frac{c^2}{h^4}\frac{I_\nu}{\nu^3}.} \quad (3.13)$$

This relation shows that, with an appropriate renormalization, I_ν/ν^3 *is the photons' distribution function.*

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Mean occupation number η

As an aid in defining the mean occupation number, we introduce the concept of the *density of states:* Consider a particle of mass m, described quantum mechanically. Suppose that the particle is known to be located in a volume $d\mathcal{V}_x$ (as observed in a specific inertial reference frame) and to have a spatial momentum in the region $d\mathcal{V}_p$ centered on $\mathbf{p}$. Suppose, further, that *the particle does not interact with any other particles or fields;* for example, ignore Coulomb interactions. (In portions of Chaps. 4 and 5, we include interactions.) Then how many single-particle quantum mechanical states[3] are available to the free particle? This question is answered most easily by constructing (in some arbitrary inertial frame) a complete set of wave functions for the particle's spatial degrees of freedom, with the wave functions (i) confined to be eigenfunctions of the momentum operator and (ii) confined to satisfy the standard periodic boundary conditions on the walls of a box with volume $d\mathcal{V}_x$. For simplicity, let the box have edge length L along each of the three spatial axes of the Cartesian spatial coordinates, so $d\mathcal{V}_x = L^3$. (This L is arbitrary and will drop out of our analysis shortly.) Then a complete set of wave functions satisfying (i) and (ii) is the set $\{\psi_{j,k,l}\}$ with

$$\psi_{j,k,l}(x, y, z) = \frac{1}{L^{3/2}} e^{i(2\pi/L)(jx+ky+lz)} e^{-i\omega t} \quad (3.14a)$$

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The basis states (3.14a) are eigenfunctions of the momentum operator $(\hbar/i)\nabla$ with momentum eigenvalues

$$p_x = \frac{2\pi\hbar}{L} j\,, \quad p_y = \frac{2\pi\hbar}{L} k\,, \quad p_z = \frac{2\pi\hbar}{L} l; \tag{3.14b}$$

correspondingly, the wave function's frequency ω has the following values in Newtonian theory **N** and relativity **R**:

$$\text{N} \quad \hbar\omega = E = \frac{\mathbf{p}^2}{2m} = \frac{1}{2m}\left(\frac{2\pi\hbar}{L}\right)^2 (j^2 + k^2 + l^2); \tag{3.14c}$$

$$\text{R} \quad \hbar\omega = \mathcal{E} = \sqrt{m^2 + \mathbf{p}^2} \to m + E \text{ in the Newtonian limit.} \tag{3.14d}$$

Equations (3.14b) tell us that the allowed values of the momentum are confined to lattice sites in 3-momentum space with one site in each cube of side $2\pi\hbar/L$. Correspondingly, the total number of states in the region $d\mathcal{V}_x d\mathcal{V}_p$ of phase space is the number of cubes of side $2\pi\hbar/L$ in the region $d\mathcal{V}_p$ of momentum space:

$$dN_{\text{states}} = \frac{d\mathcal{V}_p}{(2\pi\hbar/L)^3} = \frac{L^3 d\mathcal{V}_p}{(2\pi\hbar)^3} = \frac{d\mathcal{V}_x d\mathcal{V}_p}{h^3}. \tag{3.15}$$

This is true no matter how relativistic or nonrelativistic the particle may be.

Thus far we have considered only the particle's spatial degrees of freedom. Particles can also have an internal degree of freedom called "spin." For a particle with spin s, the number of independent spin states is

$$g_s = \begin{cases} 2s+1 & \text{if } m \neq 0 \text{ (e.g., an electron, proton, or atomic nucleus)} \\ 2 & \text{if } m = 0 \text{ and } s > 0 \text{ [e.g., a photon } (s=1) \text{ or graviton } (s=2)] \\ 1 & \text{if } m = 0 \text{ and } s = 0 \text{ (i.e., a hypothetical massless scalar particle)} \end{cases} \tag{3.16}$$

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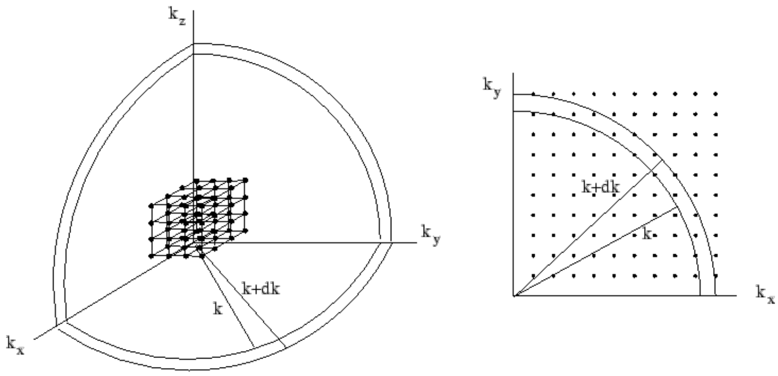
Quantum states in momentum space



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Density of states & occupation number

$$\mathcal{N}_{\text{states}} \equiv \frac{dN_{\text{states}}}{d^2\mathcal{V}} = \frac{g_s}{h^3}.$$

The ratio of the number density of particles to the number density of quantum states is obviously the number of particles in each state (the state's *occupation number*) averaged over many neighboring states—but few enough that the averaging region is small by macroscopic standards. In other words, this ratio is the quantum states' *mean occupation number* η:

$$\eta = \frac{\mathcal{N}}{\mathcal{N}_{\text{states}}} = \frac{h^3}{g_s}\mathcal{N}; \quad \text{i.e.,} \quad \boxed{\mathcal{N} = \mathcal{N}_{\text{states}}\eta = \frac{g_s}{h^3}\eta.} \tag{3.18}$$

The mean occupation number η plays an important role in quantum statistical mechanics, and its quantum roots have a profound impact on classical statistical physics.

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Fermions, bosons and the classical limit

$$0 \le \eta \le 1 \text{ for fermions}, \qquad 0 \le \eta < \infty \text{ for bosons}. \tag{3.19}$$

Quantum theory also teaches us that, when $\eta \ll 1$, the particles, whether fermions or bosons, behave like classical, discrete, distinguishable particles; and when $\eta \gg 1$ (possible only for bosons), the particles behave like a classical wave—if the particles are photons ($s = 1$), like a classical electromagnetic wave; and if they are gravitons ($s = 2$), like a classical gravitational wave. This role of η in revealing the particles' physical behavior will motivate us frequently to use η as our distribution function instead of $\mathcal{N}$.

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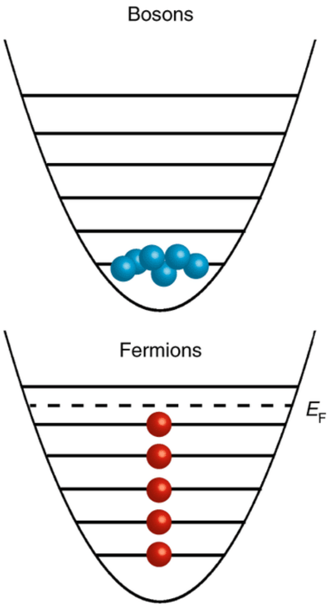
Equilibrium distribution functions (derived previously)

$$\eta = \frac{1}{e^{(E-\mu)/(k_B T)} + 1} \quad \text{for fermions,}$$

$$\eta = \frac{1}{e^{(E-\mu)/(k_B T)} - 1} \quad \text{for bosons.}$$

Notice that the equilibrium mean occupation number (3.22a) for fermions lies in the range 0–1 as required, while that (3.22b) for bosons lies in the range 0 to ∞.



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Classical or Boltzmann distribution function

The regime $\mu \ll -k_B T$, the mean occupation number is small compared to unity for all particle energies E (since E is never negative; i.e., $\mathcal{E}$ is never less than m). This is the domain of distinguishable, classical particles, and in it both the Fermi-Dirac and Bose-Einstein distributions become

$$\eta \simeq e^{-(E-\mu)/(k_B T)} = e^{-(\mathcal{E}-\tilde{\mu})/(k_B T)}$$
$$\text{when } \mu \equiv \tilde{\mu} - m \ll -k_B T \quad \text{(classical particles).}$$

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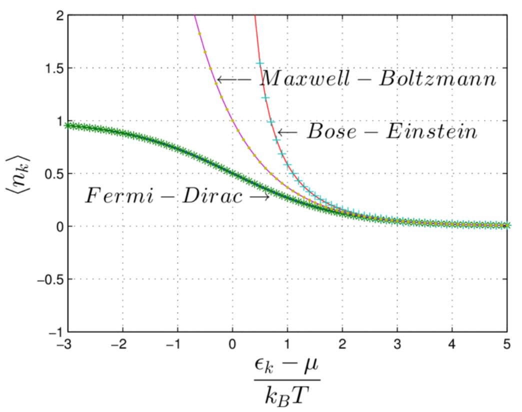
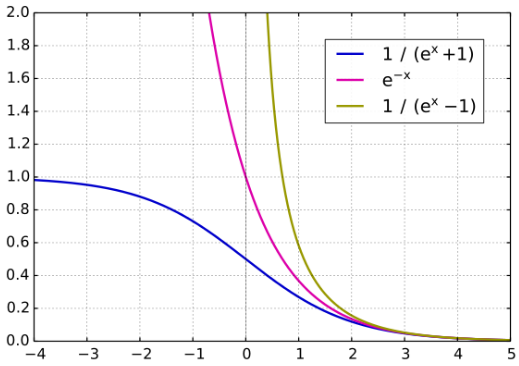
Mean occupation number: $\eta = < n_k >$





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Exercise 3.3 ***Practice and Example: Regimes of Particulate and Wave-Like Behavior* N R

(a) Cygnus X-1 is a source of X-rays that has been studied extensively by astronomers. The observations (X-ray, optical, and radio) show that it is a distance $r \sim 6{,}000$ light-years from Earth. It consists of a very hot disk of X-ray-emitting gas that surrounds a black hole with mass $15M_\odot$, and the hole in turn is in a binary orbit with a heavy companion star. Most of the X-ray photons have energies $\mathcal{E} \sim 2$ keV, their energy flux arriving at Earth is $F \sim 10^{-10}$ W m^{-2}, and the portion of the disk that emits most of them has radius roughly 7 times that of the black hole (i.e., $R \sim 300$ km).[5] Make a rough estimate of the mean occupation number of the X-rays' photon states. Your answer should be in the region $\eta \ll 1$, so the photons behave like classical, distinguishable particles. Will the occupation number change as the photons propagate from the source to Earth?

(b) A highly nonspherical supernova in the Virgo cluster of galaxies (40 million light-years from Earth) emits a burst of gravitational radiation with frequencies spread over the band 0.5–2.0 kHz, as measured at Earth. The burst comes out in a time of about 10 ms, so it lasts only a few cycles, and it carries a total energy of roughly $10^{-3}M_\odot c^2$, where $M_\odot = 2 \times 10^{30}$ kg is the mass of the Sun. The emitting region is about the size of the newly forming neutron-star core (10 km), which is small compared to the wavelength of the waves; so if one were to try to resolve the source spatially by imaging the gravitational waves with a gravitational lens, one would see only a blur of spatial size one wavelength rather than seeing the neutron star. What is the mean occupation number of the burst's graviton states? Your answer should be in the region $\eta \gg 1$, so the gravitons behave like a classical gravitational wave.

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Exercise 3.7 ***Example: Observations of Cosmic Microwave Radiation from Earth*** R T2

The universe is filled with cosmic microwave radiation left over from the big bang. At each event in spacetime the microwave radiation has a mean rest frame. As seen in that mean rest frame the radiation's distribution function η is almost precisely isotropic and thermal with zero chemical potential:

$$\eta = \frac{1}{e^{h\nu/(k_B T_o)} - 1}, \quad \text{with} \quad T_o = 2.725 \text{ K}. \tag{3.29}$$

Here ν is the frequency of a photon as measured in the mean rest frame.

(a) Show that the specific intensity of the radiation as measured in its mean rest frame has the *Planck spectrum*, Eq. (3.23). Plot this specific intensity as a function of frequency, and from your plot determine the frequency of the intensity peak.

(b) Show that η can be rewritten in the frame-independent form

$$\eta = \frac{1}{e^{-\vec{p}\cdot\vec{u}_o/(k_B T_o)} - 1}, \tag{3.30}$$

where $\vec{p}$ is the photon 4-momentum, and $\vec{u}_o$ is the 4-velocity of the mean rest frame. [Hint: See Sec. 2.6 and especially Eq. (2.29).]

(c) In actuality, Earth moves relative to the mean rest frame of the microwave background with a speed v of roughly 400 km s^{-1} toward the Hydra-Centaurus region of the sky. An observer on Earth points his microwave receiver in a direction that makes an angle θ with the direction of that motion, as measured in Earth's frame. Show that the specific intensity of the radiation received is precisely Planckian in form [Eqs. (3.23)], but with a direction-dependent *Doppler-shifted temperature*

$$T = T_o \left(\frac{\sqrt{1 - v^2}}{1 - v\cos\theta} \right). \tag{3.31}$$

Note that this Doppler shift of T is precisely the same as the Doppler shift of the frequency of any specific photon [Eq. (2.33)]. Note also that the θ dependence corresponds to an anisotropy of the microwave radiation as seen from Earth. Show that because Earth's velocity is small compared to the speed of light, the anisotropy is very nearly dipolar in form. Measurements by the WMAP satellite give $T_o = 2.725$ K and (averaged over a year) an amplitude of 3.346×10^{-3} K for the dipolar temperature variations (Bennett et al., 2003). What, precisely, is the value of Earth's year-averaged speed v?

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Particle density and flux (N)

From the definition $\mathcal{N} \equiv dN/d\mathcal{V}_x d\mathcal{V}_p$ of the distribution function, it is clear that the number density of particles $n(\mathbf{x}, t)$ in physical space is given by the integral

$$n = \frac{dN}{d\mathcal{V}_x} = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} d\mathcal{V}_p = \int \mathcal{N} d\mathcal{V}_p. \tag{3.32a}$$

Similarly, the number of particles crossing a unit surface in the y-z plane per unit time (i.e., the x component of the flux of particles) is

$$S_x = \frac{dN}{dydzdt} = \int \frac{dN}{dxdydzd\mathcal{V}_p} \frac{dx}{dt} d\mathcal{V}_p = \int \mathcal{N} \frac{p_x}{m} d\mathcal{V}_p,$$

where $dx/dt = p_x/m$ is the x component of the particle velocity. This and the analogous equations for S_y and S_z can be combined into a single geometric, coordinate-independent integral for the vectorial particle flux:

$$\mathbf{S} = \int \mathcal{N} \, \mathbf{p} \frac{d\mathcal{V}_p}{m}. \tag{3.32b}$$

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Stress tensor (N)

Notice that, if we multiply this **S** by the particles' mass m, the integral becomes the momentum density:

$$\boxed{\mathbf{G} = m\mathbf{S} = \int \mathcal{N}\mathbf{p}\, d\mathcal{V}_p.} \tag{3.32c}$$

Finally, since the stress tensor **T** is the flux of momentum [Eq. (1.33)], its j-x component (j component of momentum crossing a unit area in the y-z plane per unit time) must be

$$T_{jx} = \int \frac{dN}{dydzdtd\mathcal{V}_p} p_j\, d\mathcal{V}_p = \int \frac{dN}{dxdydzd\mathcal{V}_p}\frac{dx}{dt} p_j d\mathcal{V}_p = \int \mathcal{N} p_j \frac{p_x}{m} d\mathcal{V}_p.$$

This and the corresponding equations for T_{jy} and T_{jz} can be collected together into a single geometric, coordinate-independent integral:

$$T_{jk} = \int \mathcal{N} p_j p_k \frac{d\mathcal{V}_p}{m}, \quad \text{i.e.,} \quad \boxed{\mathbf{T} = \int \mathcal{N}\mathbf{p}\otimes\mathbf{p}\frac{d\mathcal{V}_p}{m}.} \tag{3.32d}$$

Notice that the number density n is the zeroth moment of the distribution function in momentum space [Eq. (3.32a)], and aside from factors $1/m$, the particle flux vector is the first moment [Eq. (3.32b)], and the stress tensor is the second moment [Eq. (3.32d)]. All three moments are geometric, coordinate-independent quantities, and they are the simplest such quantities that one can construct by integrating the distribution function over momentum space.

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Equations of state

If the Newtonian distribution function is isotropic in momentum space (i.e., is a function only of the magnitude $p \equiv |\mathbf{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$ of the momentum, as is the case, e.g., when the particle distribution is thermalized), then the particle flux **S** vanishes (equal numbers of particles travel in all directions), and the stress tensor is isotropic: $\mathbf{T} = P\mathbf{g}$, or $T_{jk} = P\delta_{jk}$. Thus, it is the stress tensor of a perfect fluid. [Here P is the isotropic pressure, and **g** is the metric tensor of Euclidian 3-space, with Cartesian components equal to the Kronecker delta; Eq. (1.9f).] In this isotropic case, the pressure can be computed most easily as 1/3 the trace of the stress tensor (3.32d):

$$\begin{aligned} P = \frac{1}{3}T_{jj} &= \frac{1}{3}\int \mathcal{N}(p_x^2 + p_y^2 + p_z^2)\frac{d\mathcal{V}_p}{m} \\ &= \frac{1}{3}\int_0^\infty \mathcal{N}p^2 \frac{4\pi p^2 dp}{m} = \frac{4\pi}{3m}\int_0^\infty \mathcal{N}p^4\, dp. \end{aligned} \tag{3.37a}$$

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Here in the third step we have written the momentum-volume element in spherical polar coordinates as $d\mathcal{V}_p = p^2 \sin\theta d\theta d\phi dp$ and have integrated over angles to get $4\pi p^2 dp$. Similarly, we can reexpress the number density of particles (3.32a) and the corresponding mass density as

$$n = 4\pi \int_0^\infty \mathcal{N} p^2 dp, \qquad \rho \equiv mn = 4\pi m \int_0^\infty \mathcal{N} p^2\, dp. \tag{3.37b}$$

Finally, because each particle carries an energy $E = p^2/(2m)$, the energy density in this isotropic case (which we shall denote by U) is 3/2 the pressure:

$$U = \int \frac{p^2}{2m} \mathcal{N} d\mathcal{V}_p = \frac{4\pi}{2m} \int_0^\infty \mathcal{N} p^4 dp = \frac{3}{2} P \tag{3.37c}$$

[cf. Eq. (3.37a)].

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If we know the distribution function for an isotropic collection of particles, Eqs. (3.37) give us a straightforward way of computing the collection's number density of particles n, mass density $\rho = nm$, perfect-fluid energy density U, and perfect-fluid pressure P as measured in the particles' mean rest frame. For a thermalized gas, the distribution functions (3.22a), (3.22b), and (3.22d) [with $\mathcal{N} = (g_s/h^3)\eta$] depend on two parameters: the temperature T and chemical potential μ, so this calculation gives n, U, and P in terms of μ and T. One can then invert $n(\mu, T)$ to get $\mu(n, T)$ and insert the result into the expressions for U and P to obtain *equations of state* for thermalized, nonrelativistic particles:

$$U = U(\rho, T), \quad P = P(\rho, T). \tag{3.38}$$

For a gas of nonrelativistic, classical particles, the distribution function is Boltzmann [Eq. (3.22d)], $\mathcal{N} = (g_s/h^3)e^{(\mu - E)/(k_B T)}$, with $E = p^2/(2m)$, and this procedure gives, quite easily (Ex. 3.8):

$$\boxed{n = \frac{g_s e^{\mu/(k_B T)}}{\lambda_{T\,\mathrm{dB}}^3} = \frac{g_s}{h^3}(2\pi m k_B T)^{3/2} e^{\mu/(k_B T)},} \tag{3.39a}$$

$$\boxed{U = \frac{3}{2} n k_B T, \quad P = n k_B T.} \tag{3.39b}$$

Notice that the mean energy per particle is (cf. Ex. 3.4b)

$$\bar{E} = \frac{3}{2} k_B T. \tag{3.39c}$$

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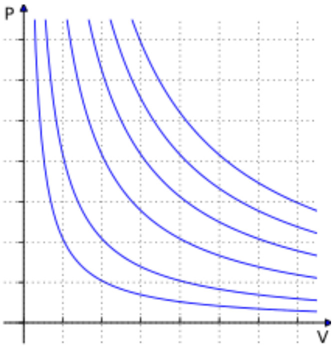
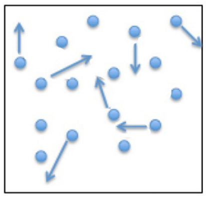
Classical ideal gas PV = Nk_BT



Propane Gas Tank



Molecules inside the gas tank

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Relativistic Number-Flux 4-Vector **S** and Stress-Energy Tensor T

When we switch from Newtonian theory to special relativity's 4-dimensional spacetime viewpoint, we require that all physical quantities be described by geometric, frame-independent objects (scalars, vectors, tensors, . . .) in 4-dimensional spacetime. We can construct such objects as momentum-space integrals over the frame-independent, relativistic distribution function $\mathcal{N}(\mathcal{P}, \vec{p}) = (g_s/h^3)\eta$. The frame-independent quantities that can appear in these integrals are (i) $\mathcal{N}$ itself, (ii) the particle 4-momentum $\vec{p}$, and (iii) the frame-independent integration element $d\mathcal{V}_p/\mathcal{E}$ [Eq. (3.7b)], which takes the form $dp_x dp_y dp_z/\sqrt{m^2 + \mathbf{p}^2}$ in any inertial reference frame. By analogy with the Newtonian regime, the most interesting such integrals are the lowest three moments of the distribution function:

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0th, 1st and 2nd moments of $\mathcal{N}$

$$R \equiv \int \mathcal{N} \frac{d\mathcal{V}_p}{\mathcal{E}};$$

$$\boxed{\vec{S} \equiv \int \mathcal{N} \vec{p} \frac{d\mathcal{V}_p}{\mathcal{E}},} \quad \text{i.e., } S^\mu \equiv \int \mathcal{N} p^\mu \frac{d\mathcal{V}_p}{\mathcal{E}};$$

$$\boxed{\boldsymbol{T} \equiv \int \mathcal{N} \vec{p} \otimes \vec{p} \frac{d\mathcal{V}_p}{\mathcal{E}},} \quad \text{i.e., } T^{\mu\nu} \equiv \int \mathcal{N} p^\mu p^\nu \frac{d\mathcal{V}_p}{\mathcal{E}}.$$

Here and throughout this chapter, relativistic momentum-space integrals are taken over the entire mass hyperboloid unless otherwise specified.

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Meaning of the moments of $\mathcal{N}$

Zeroth

$$R = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} \frac{1}{\mathcal{E}} d\mathcal{V}_p \qquad (3.34)$$

(where of course $d\mathcal{V}_x = dxdydz$ and $d\mathcal{V}_p = dp_x dp_y dp_z$). This is the sum, over all particles in a unit 3-volume, of the inverse energy. Although it is intriguing that this quantity is a frame-independent scalar, it is not a quantity that appears in any important way in the laws of physics.

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Time component of S: number density

By contrast, the 4-vector field $\vec{S}$ of Eq. (3.33b) plays a very important role in physics. Its time component in our chosen frame is

$$S^0 = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} \frac{p^0}{\mathcal{E}} d\mathcal{V}_p = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} d\mathcal{V}_p \tag{3.35a}$$

(since p^0 and $\mathcal{E}$ are just different notations for the same thing—the relativistic energy $\sqrt{m^2 + \mathbf{p}^2}$ of a particle). Obviously, this S^0 is the number of particles per unit spatial volume as measured in our chosen inertial frame:

$$S^0 = n = (\text{number density of particles}). \tag{3.35b}$$

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Spatial components of S: particle flux

The x component of $\vec{S}$ is

$$S^x = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} \frac{p^x}{\mathcal{E}} d\mathcal{V}_p = \int \frac{dN}{dx dy dz\, d\mathcal{V}_p} \frac{dx}{dt} d\mathcal{V}_p = \int \frac{dN}{dt dy dz\, d\mathcal{V}_p} d\mathcal{V}_p, \tag{3.35c}$$

which is the number of particles crossing a unit area in the y-z plane per unit time (i.e., the x component of the particle flux); similarly for other directions j:

$$S^j = (j \text{ component of the particle flux vector } \mathbf{S}). \tag{3.35d}$$

[In Eq. (3.35c), the second equality follows from

$$\frac{p^j}{\mathcal{E}} = \frac{p^j}{p^0} = \frac{dx^j/d\zeta}{dt/d\zeta} = \frac{dx^j}{dt} = (j \text{ component of velocity}), \tag{3.35e}$$

where ζ is the affine parameter such that $\vec{p} = d\vec{x}/d\zeta$.] Since S^0 is the particle number density and S^j is the particle flux, $\vec{S}$ [Eq. (3.33b)] *must be the number-flux 4-vector*

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Components of T: energy&momentum density and stress (flux of momentum)

Turn to the quantity $\boldsymbol{T}$ defined by the integral (3.33c). When we perform a 3+1 split of it in our chosen inertial frame, we find the following for its various parts:

$$T^{\mu 0} = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} p^\mu p^0 \frac{d\mathcal{V}_p}{p^0} = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} p^\mu d\mathcal{V}_p \tag{3.36a}$$

is the μ component of 4-momentum per unit volume (i.e., T^{00} is the energy density, and T^{j0} is the momentum density). Also,

$$T^{\mu x} = \int \frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} p^\mu p^x \frac{d\mathcal{V}_p}{p^0} = \int \frac{dN}{dx\,dy\,dz\,d\mathcal{V}_p} \frac{dx}{dt} p^\mu d\mathcal{V}_p = \int \frac{dN}{dt\,dy\,dz\,d\mathcal{V}_p} p^\mu d\mathcal{V}_p \tag{3.36b}$$

is the amount of μ component of 4-momentum that crosses a unit area in the y-z plane per unit time (i.e., it is the x component of flux of μ component of 4-momentum).

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More specifically, T^{0x} is the x component of energy flux (which is the same as the momentum density T^{x0}), and T^{jx} is the x component of spatial-momentum flux—or, equivalently, the jx component of the stress tensor. These and the analogous expressions and interpretations of $T^{\mu y}$ and $T^{\mu z}$ can be summarized by

$$T^{00} = (\text{energy density}), \quad T^{j0} = (\text{momentum density}) = T^{0j} = (\text{energy flux}),$$
$$T^{jk} = (\text{stress tensor}). \tag{3.36c}$$

Therefore [cf. Eq. (2.67f)], the $\boldsymbol{T}$ of Eq. (3.33c) must be the stress-energy tensor introduced and studied in Sec. 2.13. Notice that in the Newtonian limit, where $\mathcal{E} \to m$, the coordinate-independent Eq. (3.33c) for the spatial part of the stress-energy tensor (the stress) becomes $\int \mathcal{N} \mathbf{p} \otimes \mathbf{p}\, d\mathcal{V}_p/m$, which is the same as our coordinate-independent Eq. (3.32d) for the stress tensor.

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Hydrogen (p+e): Boundaries of various regimes as a function of density and temperature

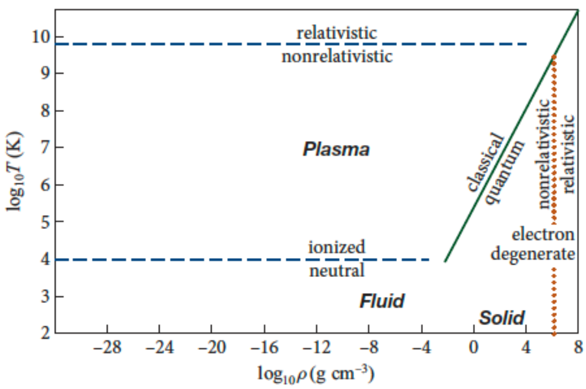

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FIGURE 3.7 Physical nature of hydrogen at various densities and temperatures. The plasma regime is discussed in great detail in Part VI, and the equation of state in this regime is Eq. (3.40). The region of relativistic electron degeneracy (to the right of the vertical dotted line) is analyzed in Sec. 3.5.4, and that for the nonrelativistic regime (between slanted solid line and vertical dotted line) in the second half of Sec. 3.5.2. The boundary between the plasma regime and the electron-degenerate regime (slanted solid line) is Eq. (3.41); that between nonrelativistic degeneracy and relativistic degeneracy (vertical dotted line) is Eq. (3.46). The upper relativistic/nonrelativistic boundary is governed by electron-positron pair production (Ex. 5.9 and Fig. 5.7) and is only crudely approximated by the upper dashed line. The ionized-neutral boundary is governed by the Saha equation (Ex. 5.10 and Fig. 20.1) and is crudely approximated by the lower dashed line. For a more accurate and detailed version of this figure, including greater detail on the plasma regime and its boundaries, see Fig. 20.1.

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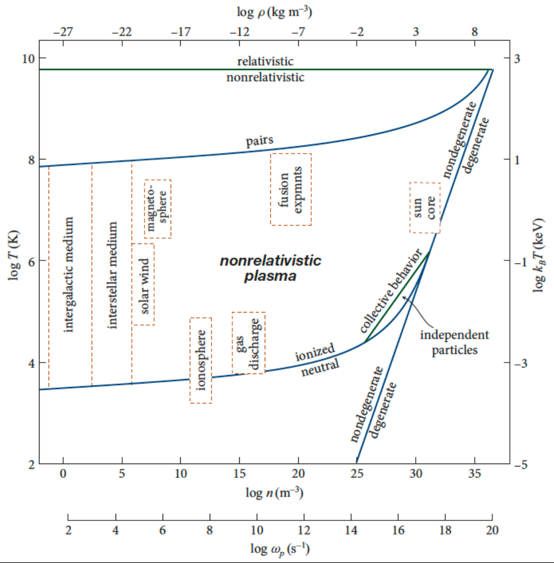
Non-relativistic plasma (chapter 20)



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Non-relativistic plasma

A nonrelativistic hydrogen plasma consists of a mixture of two fluids (gases): free electrons and free protons, in equal numbers. Each fluid has a particle number density $n = \rho/m_p$, where ρ is the total mass density and m_p is the proton mass. (The electrons are so light that they do not contribute significantly to ρ.) Correspondingly, the energy density and pressure include equal contributions from the electrons and protons and are given by [cf. Eqs. (3.39b)]

$$U = 3(k_B/m_p)\rho T , \qquad P = 2(k_B/m_p)\rho T . \qquad (3.40)$$

In zeroth approximation, the high-temperature boundary of validity for this equation of state is the temperature $T_{\rm rel} = m_e c^2/k_B = 6 \times 10^9$ K, at which the electrons become highly relativistic (top dashed line in Fig. 3.7).

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The bottom dashed line in Fig. 3.7 is the temperature $T_{\text{ion}} \sim$ (ionization energy of hydrogen)/(a few k_B) $\sim 10^4$ K, at which electrons and protons begin to recombine and form neutral hydrogen.

The solid right boundary is the point at which the electrons cease to behave like classical particles, because their mean occupation number η_e ceases to be $\ll 1$. As one can see from the Fermi-Dirac distribution (3.22a), for typical electrons (which have energies $E \sim k_B T$), the regime of classical behavior ($\eta_e \ll 1$; to the left of the solid line) is $\mu_e \ll -k_B T$ and the regime of strong quantum behavior ($\eta_e \simeq 1$; *electron degeneracy;* to the right of the solid line) is $\mu_e \gg +k_B T$. The slanted solid boundary in Fig. 3.7 is thus the location $\mu_e = 0$, which translates via Eq. (3.39a) to

$$\rho = \rho_{\text{deg}} \equiv 2m_p/\lambda_{\text{TdB}}^3 = (2m_p/h^3)(2\pi m_e k_B T)^{3/2} = 0.00808(T/10^4\ \text{K})^{3/2}\ \text{g cm}^{-3}. \quad (3.41)$$

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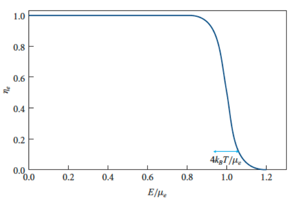
Non-relativistic degeneracy



Although the hydrogen gas is *degenerate* to the right of this boundary, we can still compute its equation of state using our kinetic-theory equations (3.37), so long as we use the quantum mechanically correct distribution function for the electrons—the Fermi-Dirac distribution (3.22a).[9] In this electron-degenerate region, $\mu_e \gg k_B T$, the electron mean occupation number $\eta_e = 1/(e^{(E-\mu_e)/(k_B T)} + 1)$ has the form shown in Fig. 3.8 and thus can be well approximated by $\eta_e = 1$ for $E = p^2/(2m_e) < \mu_e$ and $\eta_e = 0$ for $E > \mu_e$; or equivalently by

$$\eta_e = 1 \text{ for } p < p_F \equiv \sqrt{2m_e\mu_e}, \qquad \eta_e = 0 \text{ for } p > p_F. \quad (3.42)$$

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Here p_F is called the *Fermi momentum*. (The word "degenerate" refers to the fact that almost all the quantum states are fully occupied or are empty; i.e., η_e is everywhere nearly 1 or 0.) By inserting this degenerate distribution function [or, more precisely, $\mathcal{N}_e = (2/h^3)\eta_e$] into Eqs. (3.37) and integrating, we obtain $n_e \propto {p_F}^3$ and $P_e \propto {p_F}^5$. By then setting $n_e = n_p = \rho/m_p$ and solving for $p_F \propto n_e^{1/3} \propto \rho^{1/3}$ and inserting into the expression for P_e and evaluating the constants, we obtain (Ex. 3.9) the following equation of state for the electron pressure:

$$P_e = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{m_e c^2}{\lambda_c^3} \left(\frac{\rho}{m_p/\lambda_c^3}\right)^{5/3}. \quad (3.43)$$

Here

$$\lambda_c = h/(m_e c) = 2.426 \times 10^{-10}\ \text{cm} \quad (3.44)$$

is the electron Compton wavelength.

$$P = P_e = \text{Eq. (3.43)}$$

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Isotropic number density and pressure (non-relativistic)

$$n = 4\pi \int_0^\infty \mathcal{N} p^2 dp,$$

$$P = \frac{1}{3}T_{jj} = \frac{1}{3}\int \mathcal{N}(p_x^2 + p_y^2 + p_z^2)\frac{d\mathcal{V}_p}{m}$$
$$= \frac{1}{3}\int_0^\infty \mathcal{N} p^2 \frac{4\pi p^2 dp}{m} = \frac{4\pi}{3m}\int_0^\infty \mathcal{N} p^4\, dp.$$

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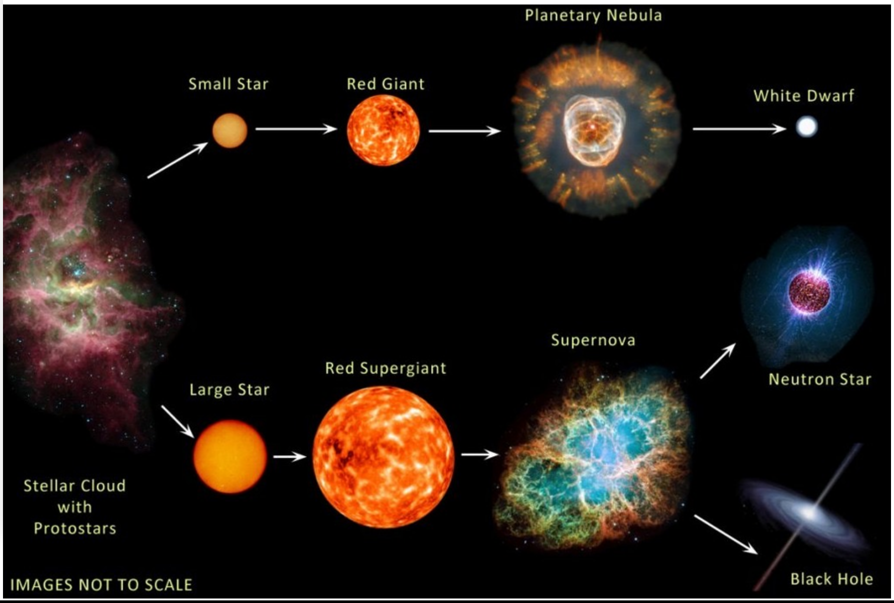
Evolution of stars



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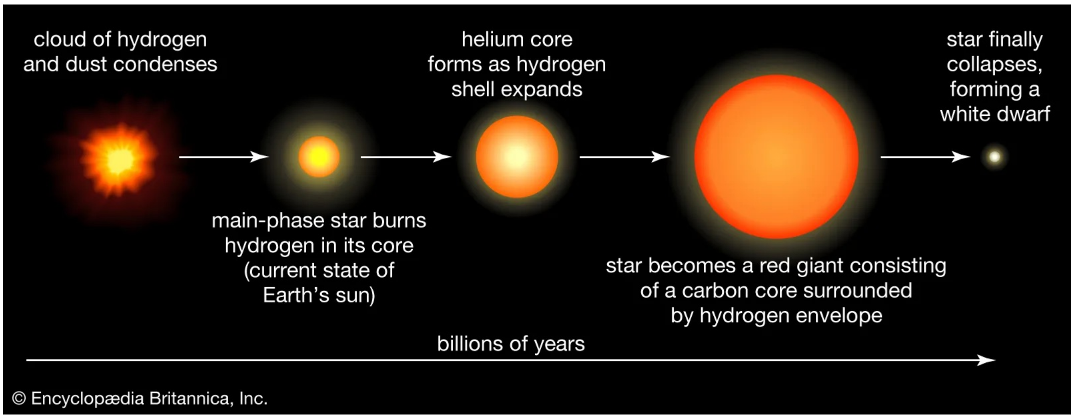
Evolution of the Sun



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White dwarfs

$$P = P_e = \text{Eq. (3.43)}$$

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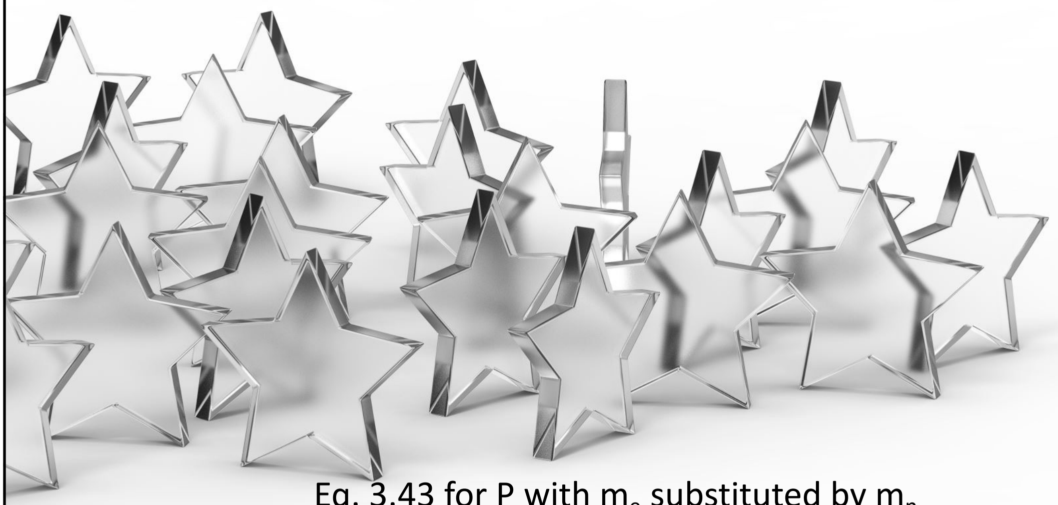
Neutron stars



Eq. 3.43 for P with m_e substituted by m_n

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Relativistic degeneracy

When the density of hydrogen in this degenerate regime is pushed on upward to

$$\rho_{\text{rel deg}} = \frac{8\pi m_p}{3\lambda_c^3} \simeq 9.8 \times 10^5 \text{ g cm}^{-3} \quad (3.46)$$

(dotted vertical line in Fig. 3.7), the electrons' zero-point motions become relativistically fast (the electron chemical potential μ_e becomes of order $m_e c^2$ and the Fermi momentum p_F of order $m_e c$), so the nonrelativistic, Newtonian analysis fails, and the matter enters a domain of relativistic degeneracy (Sec. 3.5.4). Both domains, nonrelativistic degeneracy ($\mu_e \ll m_e c^2$) and relativistic degeneracy ($\mu_e \gtrsim m_e c^2$), occur for matter inside a massive white-dwarf star—the type of star that the Sun will become

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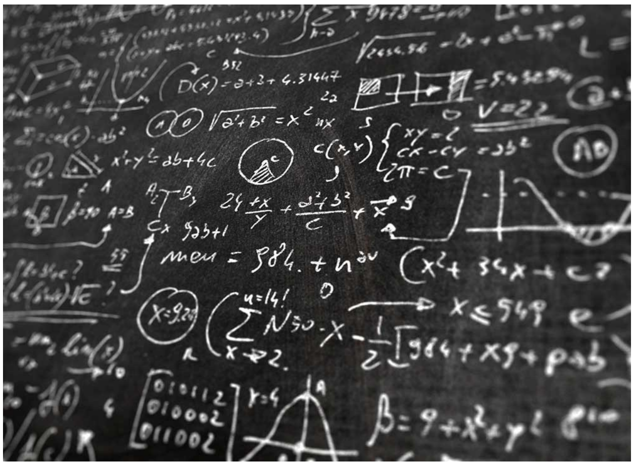
Relativistic equations of state



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Relativistic Density, Pressure, Energy Density, and Equation of State (isotropic systems)

$$\mathcal{E} = -\vec{u}_{\rm rf} \cdot \vec{p} \quad \text{expressed in frame-independent form [Eq. (2.29)],}$$
$$\mathcal{E} = p^0 = \sqrt{m^2 + p^2} \quad \text{in mean rest frame.}$$

As in Newtonian theory, isotropy greatly simplifies the momentum-space integrals (3.33) that we use to compute macroscopic properties of the particles: (i) The integrands of the expressions $S^j = \int \mathcal{N} p^j (d\mathcal{V}_p/\mathcal{E})$ and $T^{j0} = T^{0j} = \int \mathcal{N} p^j p^0 (d\mathcal{V}_p/\mathcal{E})$ for the particle flux, energy flux, and momentum density are all odd in the momentum-space coordinate p^j and therefore give vanishing integrals: $S^j = T^{j0} = T^{0j} = 0$. (ii) The integral $T^{jk} = \int \mathcal{N} p^j p^k d\mathcal{V}_p/\mathcal{E}$ produces an isotropic stress tensor, $T^{jk} = P g^{jk} = P\delta^{jk}$, whose pressure is most easily computed from its trace,

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Using these results and the relations $|\mathbf{p}| \equiv p$ for the magnitude of the momentum, $d\mathcal{V}_p = 4\pi p^2 dp$ for the momentum-space volume element, and $\mathcal{E} = p^0 = \sqrt{m^2 + p^2}$ for the particle energy, we can easily evaluate Eqs. (3.33) for the particle number density $n = S^0$, the total density of mass-energy T^{00} (which we denote ρ—the same notation as we use for mass density in Newtonian theory), and the pressure P. The results are

$$n \equiv S^0 = \int \mathcal{N} d\mathcal{V}_p = 4\pi \int_0^\infty \mathcal{N} p^2 dp,$$
$$\rho \equiv T^{00} = \int \mathcal{N} \mathcal{E} d\mathcal{V}_p = 4\pi \int_0^\infty \mathcal{N} \mathcal{E} p^2 dp,$$
$$P = \frac{1}{3} \int \mathcal{N} p^2 \frac{d\mathcal{V}_p}{\mathcal{E}} = \frac{4\pi}{3} \int_0^\infty \mathcal{N} \frac{p^4 dp}{\sqrt{m^2 + p^2}}.$$

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Equation of State for a Relativistic Degenerate Hydrogen Gas (zero T)

We can do so with the aid of the following approximation for the relativistic Fermi-Dirac mean occupation number $\eta_e = 1/[e^{(\mathcal{E}-\tilde{\mu}_e/(k_B T))} + 1]$:

$$\eta_e \simeq 1 \text{ for } \mathcal{E} < \tilde{\mu}_e \equiv \mathcal{E}_F; \text{ i.e., for } p < p_F = \sqrt{\mathcal{E}_F^2 - m^2}, \tag{3.50}$$

$$\eta_e \simeq 0 \text{ for } \mathcal{E} > \mathcal{E}_F; \text{ i.e., for } p > p_F. \tag{3.51}$$

Here $\mathcal{E}_F$ is called the relativistic *Fermi energy* and p_F the relativistic *Fermi momentum.* By inserting this η_e along with $\mathcal{N}_e = (2/h^3)\eta_e$ into the integrals (3.49) for the electron number density n_e, total density of mass-energy ρ_e, and pressure P_e, and performing the integrals (Ex. 3.10), we obtain results that are expressed most simply in terms of a parameter t (not to be confused with time) defined by

$$\mathcal{E}_F \equiv \tilde{\mu}_e \equiv m_e \cosh(t/4), \qquad p_F \equiv \sqrt{\mathcal{E}_F^2 - m_e^2} \equiv m_e \sinh(t/4). \tag{3.52a}$$

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Relativistic Degenerate Electron Gas

The results are

$$n_e = \frac{8\pi}{3\lambda_c^3}\left(\frac{p_F}{m_e}\right)^3 = \frac{8\pi}{3\lambda_c^3}\sinh^3(t/4), \tag{3.52b}$$

$$\rho_e = \frac{8\pi m_e}{\lambda_c^3}\int_0^{p_F/m_e} x^2\sqrt{1+x^2}\,dx = \frac{\pi m_e}{4\lambda_c^3}[\sinh(t) - t], \tag{3.52c}$$

$$P_e = \frac{8\pi m_e}{\lambda_c^3}\int_0^{p_F/m_e} \frac{x^4}{\sqrt{1+x^2}}\,dx = \frac{\pi m_e}{12\lambda_c^3}[\sinh(t) - 8\sinh(t/2) + 3t]. \tag{3.52d}$$

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White dwarfs

In a white-dwarf star, the protons, with their high rest mass, are nondegenerate, the total density of mass-energy is dominated by the proton rest-mass density, and since there is one proton for each electron in the hydrogen gas, that total is

$$\rho \simeq m_p n_e = \frac{8\pi m_p}{3\lambda_c^3} \sinh^3(t/4). \qquad (3.53a)$$

By contrast (as in the nonrelativistic regime), the pressure is dominated by the electrons (because of their huge zero-point motions), not the protons; and so the total pressure is

$$P = P_e = \frac{\pi\, m_e}{12\lambda^3}[\sinh(t) - 8\sinh(t/2) + 3t]. \qquad (3.53b)$$

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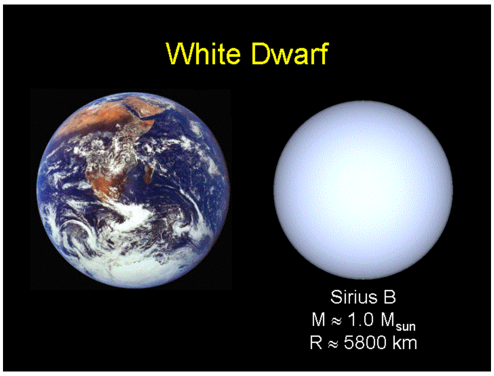

In the low-density limit, where $t \ll 1$ so $p_F \ll m_e = m_e c$, we can solve the relativistic equation (3.52b) for t as a function of $n_e = \rho/m_p$ and insert the result into the relativistic expression (3.53b); the result is the nonrelativistic equation of state (3.43).

The dividing line $\rho = \rho_{\rm rel\,deg} = 8\pi m_p/(3\lambda_c^3) \simeq 1.0 \times 10^6$ g cm^{-3} [Eq. (3.46)] between nonrelativistic and relativistic degeneracy is the point where the electron Fermi momentum is equal to the electron rest mass [i.e., $\sinh(t/4) = 1$]. The equation of state (3.53a) and (3.53b) implies

$$P_e \propto \rho^{5/3} \quad \text{in the nonrelativistic regime, } \rho \ll \rho_{\rm rel\,deg},$$
$$P_e \propto \rho^{4/3} \quad \text{in the relativistic regime, } \rho \gg \rho_{\rm rel\,deg}. \qquad (3.53c)$$

These asymptotic equations of state turn out to play a crucial role in the structure and stability of white dwarf stars

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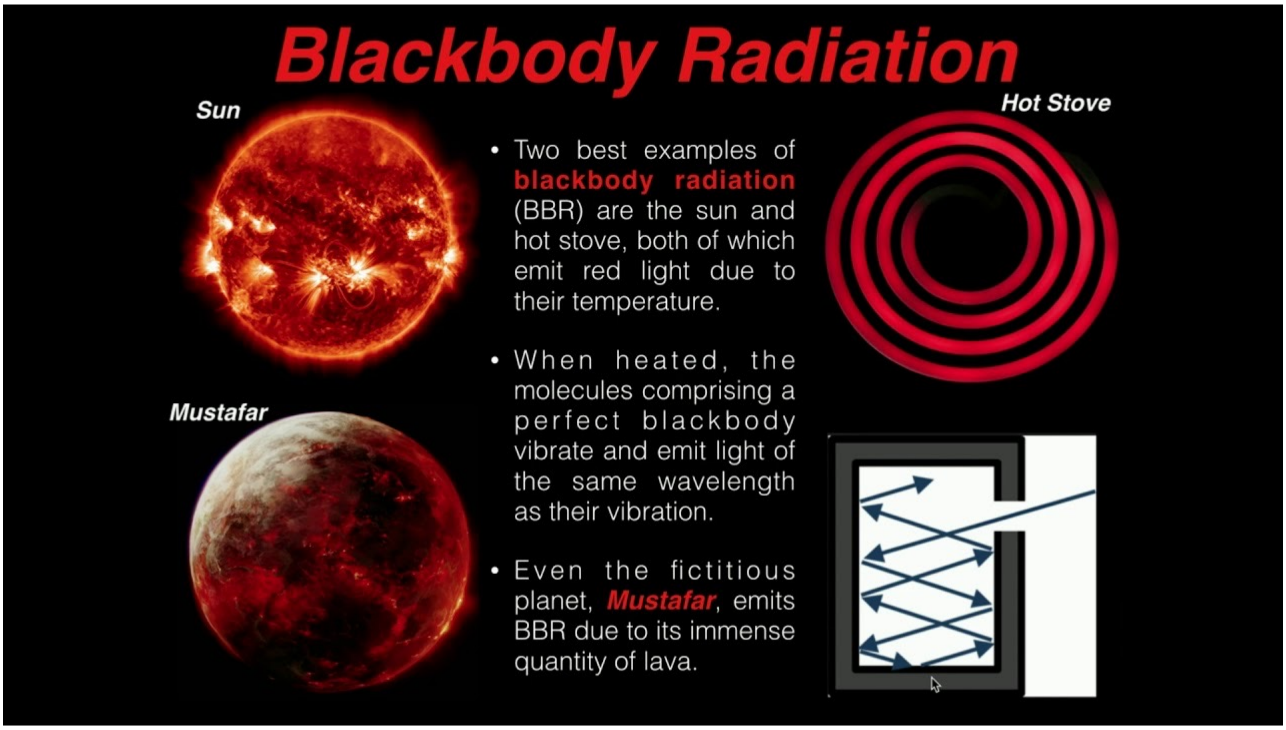

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Equation of state for termal radiation

As was discussed at the end of Sec. 3.3, for a gas of thermalized photons in an environment where photons are readily created and absorbed, the distribution function has the blackbody (Planck) form $\eta = 1/(e^{\mathcal{E}/(k_B T)} - 1)$, which we can rewrite as $1/(e^{p/(k_B T)} - 1)$, since the energy $\mathcal{E}$ of a photon is the same as the magnitude p of its momentum. In this case, the relativistic integrals (3.49) give (see Ex. 3.13)

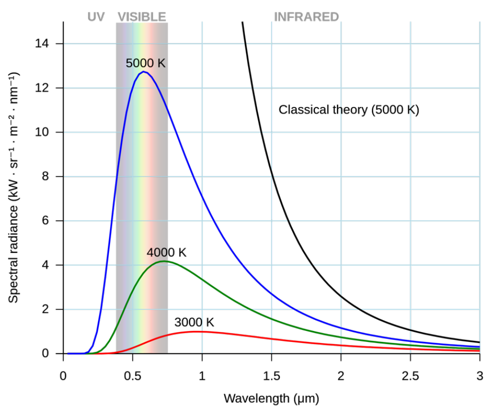
$$\boxed{n = bT^3, \qquad \rho = aT^4, \qquad P = \frac{1}{3}\rho,} \tag{3.54a}$$

where

$$b = 16\pi\zeta(3)\frac{k_B^3}{h^3c^3} = 20.28\ \mathrm{cm^{-3}\,K^{-3}}, \tag{3.54b}$$

$$a = \frac{8\pi^5}{15}\frac{k_B^4}{h^3c^3} = 7.566 \times 10^{-15}\ \mathrm{erg\,cm^{-3}\,K^{-4}} = 7.566 \times 10^{-16}\ \mathrm{J\,m^{-3}\,K^{-4}} \tag{3.54c}$$

are *radiation constants*. Here $\zeta(3) = \sum_{n=1}^{\infty} n^{-3} = 1.2020569\ldots$ is the Riemann zeta function.



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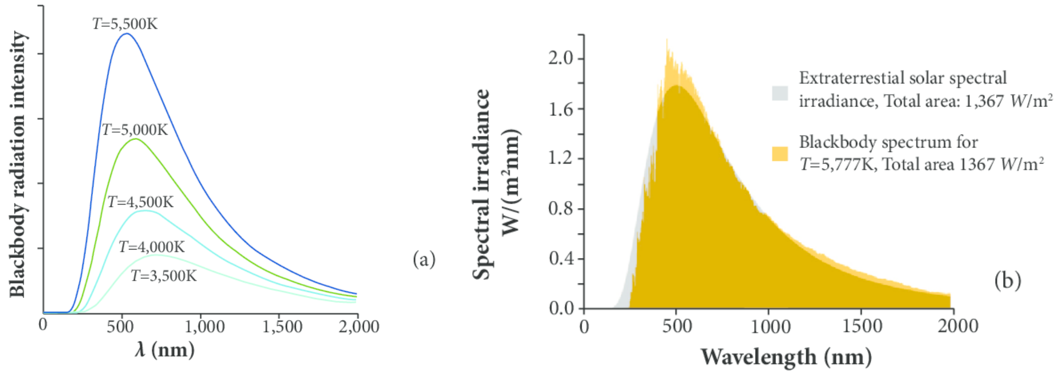
Comparison of blackbody radiation to the solar irradiance



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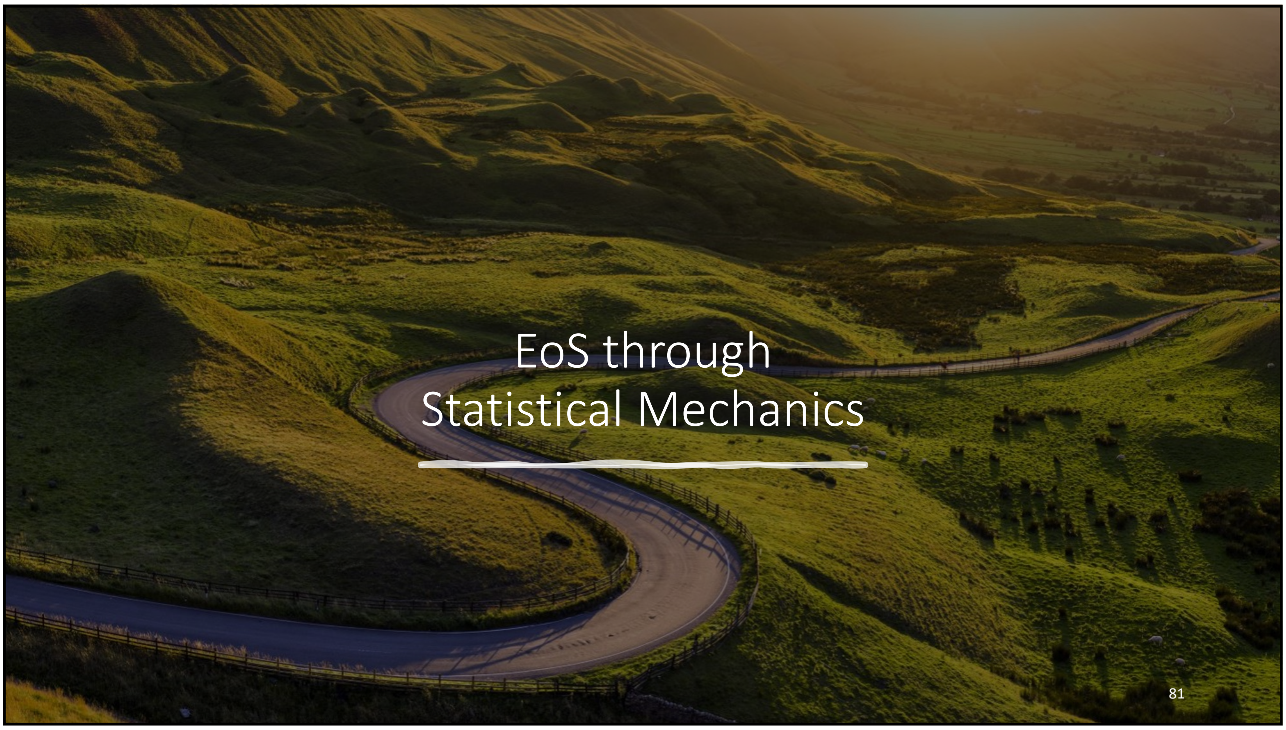

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Quantum gases: fermions and bosons (grand canonical ensemble)

The equation of state for a quantum ideal gas is

$$pV = kT\ln\Xi = \pm kT\sum_j ln[1 \pm e^{\beta\mu}e^{-\beta\varepsilon_j}]$$

The summation over states can be replaced by an integration over energy levels with:

$$\omega(\varepsilon)d\varepsilon = 2\pi\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} V\varepsilon^{\frac{1}{2}}d\varepsilon \qquad \text{3D}$$

From this, derive the quantum virial expansion (where $\lambda = e^{\beta\mu}$):

$$\frac{P}{kT} = \mp\frac{1}{\Lambda^3}\sum_{j=1}^{\infty}\frac{(\mp1)^j\lambda^j}{j^{\frac{5}{2}}}$$

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EoS of Bosons for any dimension $\mathcal{D}$

$$D(\epsilon) = \frac{V}{\Gamma(\mathcal{D}/2)}\left(\frac{m}{2\pi\hbar^2}\right)^{\mathcal{D}/2}\epsilon^{\mathcal{D}/2-1}, \quad V = L^{\mathcal{D}}.$$

Fundamental thermodynamic relations for BE gas:

$$\frac{pV}{k_BT} = -\sum_k \ln\left(1 - ze^{-\beta\epsilon_k}\right) = -\int_0^\infty d\epsilon\, D(\epsilon)\ln\left(1 - ze^{-\beta\epsilon}\right) = \frac{V}{\lambda_T^{\mathcal{D}}}g_{\mathcal{D}/2+1}(z),$$

$$\mathcal{N} = \sum_k \frac{1}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon\,\frac{D(\epsilon)}{z^{-1}e^{\beta\epsilon} - 1} = \frac{V}{\lambda_T^{\mathcal{D}}}g_{\mathcal{D}/2}(z), \quad z < 1,$$

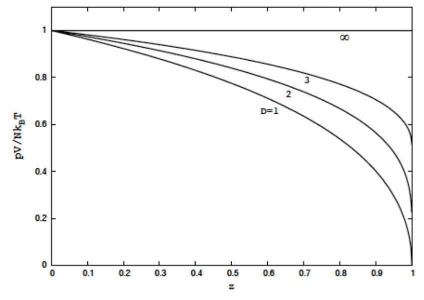
$$U = \sum_k \frac{\epsilon_k}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon\,\frac{D(\epsilon)\epsilon}{z^{-1}e^{\beta\epsilon} - 1} = \frac{\mathcal{D}}{2}k_BT\frac{V}{\lambda_T^{\mathcal{D}}}g_{\mathcal{D}/2+1}(z).$$

The range of fugacity is limited to the interval $0 \le z \le 1$. At $z = 1$, the expression for $\mathcal{N}$ must, in some cases, be amended by an additive term to account for the possibility of a macroscopic population of the lowest energy level (at $\epsilon = 0$).

where $z = e^{\beta\mu}$

Equation of state (with fugacity z in the role of parameter):

$$\frac{pV}{\mathcal{N}k_BT} = \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}, \quad z < 1.$$



Note the classical limit when the dimensionality $\mathcal{D} \to \infty$

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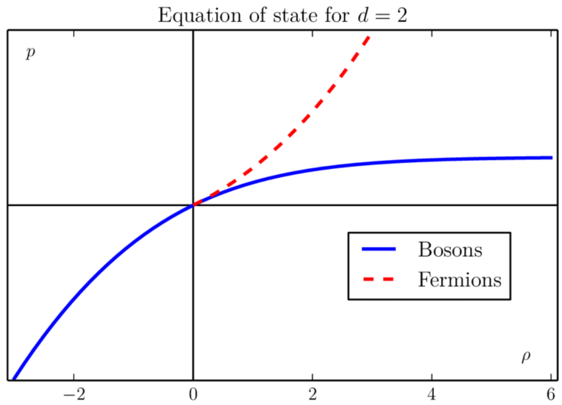
EoS for Bosons and Fermions in 2$\mathcal{D}$

- The classical limit is the straight line for positive *p* below the FD and above the BE equations.
- The effective repulsions in FD increase *p* while the effective attractions in BE decrease it, w. r. to the classical EoS.



Exercise 3.10 *Derivation and Practice: Equation of State for Relativistic, Electron-Degenerate Hydrogen* R T2
Derive the equations of state (3.52) for an electron-degenerate hydrogen gas. (Note: It might be easiest to compute the integrals with the help of symbolic manipulation software, such as Mathematica, Matlab, or Maple.)

Exercise 3.12 *Example: Specific Heat for Phonons in an Isotropic Solid* N

In Sec. 12.2 we will study classical sound waves propagating through an isotropic, elastic solid. As we shall see, there are two types of sound waves: *longitudinal* with frequency-independent speed C_L, and *transverse* with a somewhat smaller frequency-independent speed C_T. For each type of wave, $s = L$ or T, the material of the solid undergoes an elastic displacement $\boldsymbol{\xi} = A\mathbf{f}_s \exp(i\mathbf{k}\cdot\mathbf{x} - \omega t)$, where A is the wave amplitude, $\mathbf{f}_s$ is a unit vector (polarization vector) pointing in the direction of the displacement, $\mathbf{k}$ is the wave vector, and ω is the wave frequency. The wave speed is $C_s = \omega/|\mathbf{k}|$ ($= C_L$ or C_T). Associated with these waves are quanta called phonons. As for any wave, each phonon has a momentum related to its wave vector by $\mathbf{p} = \hbar\mathbf{k}$, and an energy related to its frequency by $E = \hbar\omega$. Combining these relations we learn that the relationship between a phonon's energy and the magnitude $p = |\mathbf{p}|$ of its momentum is $E = C_s p$. This is the same relationship as for photons, but with the speed of light replaced by the speed of sound! For longitudinal waves $\mathbf{f}_L$ is in the propagation direction $\mathbf{k}$, so there is just one polarization, $g_L = 1$. For transverse waves $\mathbf{f}_T$ is orthogonal to $\mathbf{k}$, so there are two orthogonal polarizations (e.g., $\mathbf{f}_T = \mathbf{e}_x$ and $\mathbf{f}_T = \mathbf{e}_y$ when $\mathbf{k}$ points in the $\mathbf{e}_z$ direction), $g_T = 2$.

(a) Phonons of both types, longitudinal and transverse, are bosons. Why? [Hint: Each normal mode of an elastic body can be described mathematically as a harmonic oscillator.]

(b) Phonons are fairly easily created, absorbed, scattered, and thermalized. A general argument that we will give for chemical reactions in Sec. 5.5 can be applied to phonon creation and absorption to deduce that, once they reach complete thermal equilibrium with their environment, the phonons will have vanishing chemical potential $\mu = 0$. What, then, will be their distribution functions η and $\mathcal{N}$?

(c) Ignoring the fact that the sound waves' wavelengths $\lambda = 2\pi/|\mathbf{k}|$ cannot be smaller than about twice the spacing between the atoms of the solid, show that the total phonon energy (wave energy) in a volume V of the solid is identical to that for blackbody photons in a volume V, but with the speed of light c replaced by the speed of sound C_s, and with the photon number of spin states, 2, replaced by $g_s = 3$ (2 for transverse waves plus 1 for longitudinal): $E_{\text{tot}} = a_s T^4 V$, with $a_s = g_s(4\pi^5/15)(k_B^4/(h^3 C_s^3))$ [cf. Eqs. (3.54)].

(d) Show that the specific heat of the phonon gas (the sound waves) is $C_V = 4a_s T^3 V$. This scales as T^3, whereas in a metal the specific heat of the degenerate electrons scales as T (previous exercise), so at sufficiently low temperatures the electron specific heat will dominate over that of the phonons.

(e) Show that in the phonon gas, only phonon modes with wavelengths longer than $\sim\lambda_T = C_s h/(k_B T)$ are excited; that is, for $\lambda \ll \lambda_T$ the mean occupation number is $\eta \ll 1$; for $\lambda \sim \lambda_T$, $\eta \sim 1$; and for $\lambda \gg \lambda_T$, $\eta \gg 1$. As T is increased, λ_T gets reduced. Ultimately it becomes of order the interatomic spacing, and our computation fails, because most of the modes that our calculation assumes are thermalized actually don't exist. What is the critical temperature (*Debye temperature*) at which our computation fails and the T^3 law for C_V changes? Show by a roughly one-line argument that above the Debye temperature, C_V is independent of temperature.

Formalism: Liouville's Theorem

The foundation for the collision-free evolution law will be *Liouville's theorem*. Consider a set $\mathcal{S}$ of particles that are initially all near some location in phase space and initially occupy an infinitesimal (frame-independent) phase-space volume $d^2\mathcal{V} = d\mathcal{V}_x d\mathcal{V}_p$. Pick a particle at the center of the set $\mathcal{S}$ and call it the "fiducial particle." Since all the particles in $\mathcal{S}$ have nearly the same initial position and velocity, they subsequently all move along nearly the same trajectory (world line): they all remain congregated around the fiducial particle. Liouville's theorem says that the phase-space volume occupied by the set of particles $\mathcal{S}$ is conserved along the trajectory of the fiducial particle:

$$\frac{d}{d\ell}(d\mathcal{V}_x d\mathcal{V}_p) = 0. \qquad (3.63)$$

Here ℓ is an arbitrary parameter along the trajectory. For example, in Newtonian theory ℓ could be universal time t or distance l traveled, and in relativity it could be proper time τ as measured by the fiducial particle (if its rest mass is nonzero) or the affine parameter ζ that is related to the fiducial particle's 4-momentum by $\vec{p} = d\vec{x}/d\zeta$.

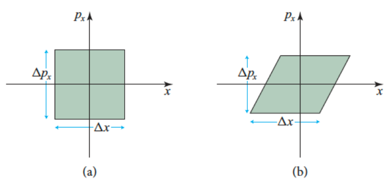


FIGURE 3.9 The phase-space region (x-p_x part) occupied by a set $\mathcal{S}$ of particles with finite rest mass, as seen in the inertial frame of the central, fiducial particle. (a) The initial region. (b) The region after a short time.

Proof

We shall prove Liouville's theorem with the aid of the diagrams in Fig. 3.9. Assume, for simplicity, that the particles have nonzero rest mass. Consider the region in phase space occupied by the particles, as seen in the inertial reference frame (rest frame) of the fiducial particle, and choose for ℓ the time t of that inertial frame (or in Newtonian theory the universal time t). Choose the particles' region $d\mathcal{V}_x d\mathcal{V}_p$ at $t = 0$ to be a rectangular box centered on the fiducial particle (i.e., on the origin $x^j = 0$ of its inertial frame; Fig. 3.9a). Examine the evolution with time t of the 2-dimensional slice $y = p_y = z = p_z = 0$ through the occupied region. The evolution of other slices will be similar. Then, as t passes, the particle at location (x, p_x) moves with velocity $dx/dt = p_x/m$ (where the nonrelativistic approximation to the velocity is used, because all the particles are very nearly at rest in the fiducial particle's inertial frame). Because the particles move freely, each has a conserved p_x, and their motion $dx/dt = p_x/m$ (larger speeds are higher in the diagram) deforms the particles' phase space region into a skewed parallelogram as shown in Fig. 3.9b. Obviously, the area of the occupied region, $\Delta x \Delta p_x$, is conserved.

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Collisionless Boltzmann equation

Since, in the absence of collisions or other nongravitational interactions, the number dN of particles in the set $\mathcal{S}$ is conserved, Liouville's theorem immediately implies the conservation of the number density in phase space, $\mathcal{N} = dN/(d\mathcal{V}_x d\mathcal{V}_p)$:

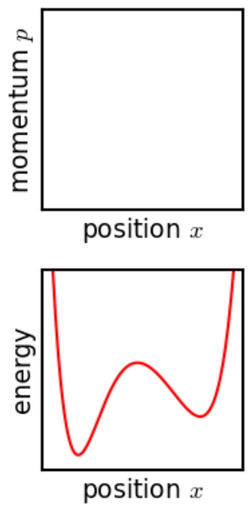
$$\boxed{\frac{d\mathcal{N}}{d\ell} = 0 \quad \text{along the trajectory of a fiducial particle.}} \tag{3.64}$$

This conservation law is called the *collisionless Boltzmann equation;* in the context of plasma physics (Part VI) it is sometimes called the *Vlasov equation.* Note that it says that *not only is the distribution function $\mathcal{N}$ frame independent; $\mathcal{N}$ also is constant along the phase-space trajectory of any freely moving particle.*

The collisionless Boltzmann equation is most nicely expressed in the frame-independent form Eq. (3.64). For some purposes, however, it is helpful to express the equation in a form that relies on a specific but arbitrary choice of inertial reference frame. Then $\mathcal{N}$ can be regarded as a function of the reference frame's seven phase-space coordinates, $\mathcal{N} = \mathcal{N}(t, x^j, p_k)$, and the collisionless Boltzmann equation (3.64) takes the coordinate-dependent form

$$\frac{d\mathcal{N}}{d\ell} = \frac{dt}{d\ell}\frac{\partial \mathcal{N}}{\partial t} + \frac{dx_j}{d\ell}\frac{\partial \mathcal{N}}{\partial x_j} + \frac{dp_j}{d\ell}\frac{\partial \mathcal{N}}{\partial p_j} = \frac{dt}{d\ell}\left(\frac{\partial \mathcal{N}}{\partial t} + v_j \frac{\partial \mathcal{N}}{\partial x_j}\right) = 0. \tag{3.65}$$

Here we have used the equation of straight-line motion $dp_j/dt = 0$ for the particles and have set dx_j/dt equal to the particle velocity v_j.



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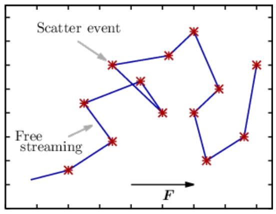
Boltzmann transport equation

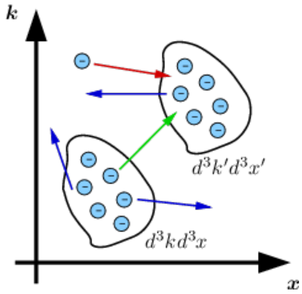
Since our derivation of the collisionless Boltzmann equation relies on the assumption that no particles are created or destroyed as time passes, the collisionless Boltzmann equation in turn should guarantee conservation of the number of particles, $\partial n/\partial t + \nabla \cdot \mathbf{S} = 0$ in Newtonian theory (Sec. 1.8), and $\vec{\nabla} \cdot \vec{S} = 0$ relativistically (Sec. 2.12.3). Indeed, this is so; see Ex. 3.14. Similarly, since the collisionless Boltzmann equation is based on the law of momentum (or 4-momentum) conservation for all the individual particles, it is reasonable to expect that the collisionless Boltzmann equation will guarantee the conservation of their total Newtonian momentum $[\partial \mathbf{G}/\partial t + \nabla \cdot \mathbf{T} = 0$, Eq. (1.36)] and their relativistic 4-momentum $[\vec{\nabla} \cdot \boldsymbol{T} = 0$, Eq. (2.73a)]. And indeed, these conservation laws do follow from the collisionless Boltzmann equation; see Ex. 3.14.

Thus far we have assumed that the particles move freely through phase space with no collisions. If collisions occur, they will produce some nonconservation of $\mathcal{N}$ along the trajectory of a freely moving, noncolliding fiducial particle, and correspondingly, the collisionless Boltzmann equation will be modified to read

$$\frac{d\mathcal{N}}{d\ell} = \left(\frac{d\mathcal{N}}{d\ell}\right)_{\text{collisions}}, \tag{3.66}$$

where the right-hand side represents the effects of collisions. This equation, with collision terms present, is called the *Boltzmann transport equation*. The actual form of the collision terms depends, of course, on the details of the collisions. We meet some specific examples in the next section [Eqs. (3.79), (3.86a), (3.87), and Ex. 3.21] and in our study of plasmas (Chaps. 22 and 23).





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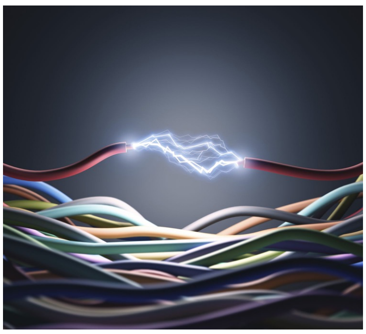
Transport coefficients



What are transport coefficients? An example is electrical conductivity κ_e. When an electric field E is imposed on a sample of matter, Ohm's law tells us that the matter responds by developing a current density

$$\mathbf{j} = \kappa_e \mathbf{E}. \tag{3.70a}$$

The electrical conductivity is high if electrons can move through the material with ease; it is low if electrons have difficulty moving. The impediment to electron motion is scattering off other particles—off ions, other electrons, phonons (sound waves), plasmons (plasma waves), ... Ohm's law is valid when (as almost always) the electrons scatter many times, so they *diffuse* (random-walk their way) through the material. To compute the electrical conductivity, one must analyze, statistically, the effects of the many scatterings on the electrons' motions. The foundation for an accurate analysis is the Boltzmann transport equation (3.66).

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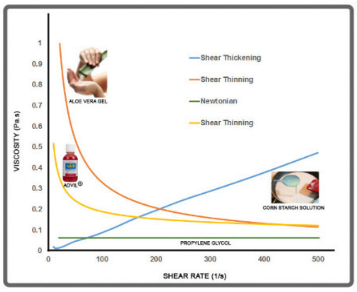
Transport coefficients



Another example of a transport coefficient is thermal conductivity κ, which appears in the law of heat conduction

$$\boxed{\mathbf{F} = -\kappa \nabla T.} \tag{3.70b}$$

Here $\mathbf{F}$ is the diffusive energy flux from regions of high temperature T to low. The impediment to heat flow is scattering of the conducting particles; and, correspondingly, the foundation for accurately computing κ is the Boltzmann transport equation.

Other examples of transport coefficients are (i) the coefficient of shear viscosity $\eta_{\rm shear}$, which determines the stress T_{ij} (diffusive flux of momentum) that arises in a shearing fluid [Eq. (13.68)]

$$\boxed{T_{ij} = -2\eta_{\rm shear}\sigma_{ij},} \tag{3.70c}$$

where σ_{ij} is the fluid's rate of shear (Ex. 3.19), and (ii) the diffusion coefficient D, which determines the diffusive flux of particles $\mathbf{S}$ from regions of high particle density n to low (Fick's law):

$$\boxed{\mathbf{S} = -D\nabla n.} \tag{3.70d}$$

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Transport coefficients

There is a *diffusion equation* associated with each of these transport coefficients. For example, the differential law of particle conservation $\partial n/\partial t + \nabla \cdot \mathbf{S} = 0$ [Eq. (1.30)], when applied to material in which the particles scatter many times so $\mathbf{S} = -D\nabla n$, gives the following diffusion equation for the particle number density:

$$\boxed{\frac{\partial n}{\partial t} = D\nabla^2 n,} \tag{3.71}$$

where we have assumed that D is spatially constant. In Ex. 3.17, by exploring solutions to this equation, we shall see that the root mean square (rms) distance $\bar{l}$ the particles travel is proportional to the square root of their travel time, $\bar{l} = \sqrt{4Dt}$, a behavior characteristic of diffusive random walks.[10] See Sec. 6.3 for deeper insights into this.

Similarly, the law of energy conservation, when applied to diffusive heat flow $\mathbf{F} = -\kappa\nabla T$, leads to a diffusion equation for the thermal energy density U and thence for temperature [Ex. 3.18 and Eq. (18.4)]. Maxwell's equations in a magnetized fluid, when combined with Ohm's law $\mathbf{j} = \kappa_e \mathbf{E}$, lead to diffusion equation (19.6) for magnetic field lines. And the law of angular momentum conservation, when applied to a shearing fluid with $T_{ij} = -2\eta_{\rm shear}\sigma_{ij}$, leads to diffusion equation (14.6) for vorticity.

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Transport coefficients

These diffusion equations, and all other physical laws involving transport coefficients, are approximations to the real world—approximations that are valid if and only if (i) many particles are involved in the transport of the quantity of interest (e.g., charge, heat, momentum, particles) and (ii) on average each particle undergoes many scatterings in moving over the length scale of the macroscopic inhomogeneities that drive the transport. This second requirement can be expressed quantitatively in terms of the *mean free path* λ between scatterings (i.e., the mean distance a particle travels between scatterings, as measured in the mean rest frame of the matter) and the *macroscopic inhomogeneity scale* $\mathcal{L}$ for the quantity that drives the transport (e.g., in heat transport that scale is $\mathcal{L} \sim T/|\nabla T|$; i.e., it is the scale on which the temperature changes by an amount of order itself). In terms of these quantities, the second criterion of validity is $\lambda \ll \mathcal{L}$. These two criteria (many particles and $\lambda \ll \mathcal{L}$) together are called *diffusion criteria,* since they guarantee that the quantity being transported (charge, heat, momentum, particles) will diffuse through the matter. If either of the two diffusion criteria fails, then the standard transport law (Ohm's law, the law of heat conduction, the Navier-Stokes equation, or the particle diffusion equation) breaks down and the corresponding transport coefficient becomes irrelevant and meaningless.

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