UNIVERSO PRIMITIVO

Mestrado em Física Astronomia 2025-2026

Exercise Sheet 3

1. As derived in class, the entropy of the primordial fluid can be defined as $S = V\,(\rho + P)/T$. Use the continuity equation to show that this definition leads to the conservation of entropy [**Hint**: prove that the time derivative of S is zero].

2. According to the standard model of particle physics, photons, electrons, neutrinos and their antiparticles are the main relativistic thermal species in the temperature plateau $1 \leq T/\mathrm{MeV} \leq 30$.
   2.1. Compute the number and energy densities of primordial photons at $T = 1\mathrm{MeV}$, in cgs units. Compare your findings with present CMB observations of these quantities: $n_{\gamma,0} \simeq 410\ \mathrm{cm}^{-3}$; $\rho_{\gamma,0} \simeq 4 \times 10^{-34}\ \mathrm{gcm}^{-3}$.
   2.2. Compute the effective number of relativistic species in energy and entropy (g_* and g_{*S}) in the temperature plateau from 1 to 30 MeV. What would be the value of g_* and g_{*S} in a particle physics model with 4 families of massless neutrinos, all with just one helicity state?
   2.3. Derive exact expressions for the plasma temperature, energy density and entropy density, $s = S/V$, as a function of redshift, z, using the condition $T_i = 1\ \mathrm{MeV}$ at $z_i = 6 \times 10^9$ as the normalization for the temperature-redshift relation.

3. Consider the Friedmann equation written as in exercise 3.2 of problems sheet 1.
   3.1. Explain why the energy density of relativistic particles (the radiation term in this equation) should be modified to:

$$\Delta_R(a)\ \Omega_{r0} \left(\frac{a_0}{a}\right)^4, \qquad \text{where:} \qquad \Delta_R(a) = \frac{g_*(a)}{g_*(a_0)} \left(\frac{g_{*S}(a_0)}{g_{*S}(a)}\right)^{4/3}.$$

   3.2. Compute the age of the universe by the end of the Big Bang Nucleosynthesis, $T = 0.1$ MeV, assuming the following approximation for g_* and g_{*S}:

$$g_* \simeq g_{*S} \simeq \begin{cases} 100 & T > 300\ \mathrm{MeV} \\ 10 & 300\ \mathrm{MeV} > T > 1\ \mathrm{MeV} \\ 3 & T <\ \mathrm{MeV} \end{cases}$$

   Use the present-day values $H_0 \simeq 1.44 \times 10^{-42}$ GeV, $\Omega_{r0} \simeq 9.2 \times 10^{-5}$. Consider that $T = 0.1\mathrm{MeV}$ at $z = 4 \times 10^8$, for the normalisation of the temperature – z relation.
   3.3. Repeat the calculation now using the tabulated values of g_* and g_{*S} in Ref. astro-ph/1609.04979 (Table A1). Compare with your findings in 3.2.

4. Read sections 3.1 and 3.2 in Ref. astro-ph/1808.08968 where the authors discuss the effect of extra-degrees of freedom in the spectrum of gravitational waves (GW) from a network of cosmic strings. Explain by your own words their findings in Fig. 6 and say if the LISA space mission would be able to discriminate between the models they investigate.

5. Consider a particle species that remains relativistic after it decouples from the primordial fluid. Derive how the temperature and number density of this species scale with redshift after decoupling. Now consider instead a particle species that decouples while it is non-relativistic. How do its temperature and number density scale with redshift in this case?
   [*Hint*: Assume that no particles of the species are created or destroyed after decoupling. Study how momentum scales with the scale factor to show that the shape of the species' distribution function does not change after decoupling]