Basic definitions and conventions

Rui Santos

FCUL & CFTC

2026

Natural units

We set:

$$\hbar = c = 1$$

- Reduces the number of fundamental constants
- Avoids ambiguities related to units
- Emphasises relativistic structure

Consequences of Natural Units

$$E = \hbar\omega = \hbar c \kappa \quad \Rightarrow \quad E = \kappa$$

$$E^2 = p^2 + m^2$$

$$E = \frac{1}{2} m v^2 \quad \text{(non-relativistic)}$$

Dimensional Analysis

- Energy, momentum and mass have the same dimensions
- Length has dimension $[E]^{-1}$

$$[E] = [p] = [m]$$

$$[x] = [t] = [E]^{-1}$$

Important

Always keep track of dimensions, even in natural units.

Basic Definitions

Four-derivatives:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

Explicitly:

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right), \qquad \partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right)$$

Metric signature:

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

Fourier Transforms in Field Theory

Spacetime $\leftrightarrow$ Momentum space

Spatial frequency k and temporal frequency ω form

$$k^\mu = (\omega, \mathrm{k})$$

Using $E = \hbar\omega$, $\mathrm{p} = \hbar\mathrm{k}$ and $\hbar = 1$:

$$k^\mu \equiv p^\mu$$

Four-dimensional Fourier transform

$$\tilde{f}(k) = \int d^4x \, e^{ik\cdot x} f(x)$$

where

$$\int d^4x = \int dx^0 dx^1 dx^2 dx^3$$

Inverse transform

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\tilde{f}(k)$$

Equivalent form:

$$\tilde{f}(\omega, \mathrm{k}) = \int d^3x\, dt\; e^{i(\omega t - \mathrm{k}\cdot\mathrm{x})} f(t, \mathrm{x})$$

Each d^4k comes with $(2\pi)^{-4}$

Example - The Dirac Delta Function

Definition

The Dirac delta function $\delta(x)$ is localised at the origin and has unit integral:

$$\int d^d x\, \delta^{(d)}(x) = 1$$

It is defined through its action on test functions:

$$\int d^d x\, f(x)\, \delta^{(d)}(x) = f(0)$$

Fourier transform

$$\tilde{\delta}^{(d)}(k) = \int d^d x \, e^{ik\cdot x} \, \delta^{(d)}(x) = 1$$

Inverse Fourier representation (in $d = 4$)

$$\int \frac{d^4 k}{(2\pi)^4} \, e^{-ik\cdot x} = \delta^{(4)}(x)$$

$$\boxed{\delta^{(4)}(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik\cdot x}}$$

Fourier Representation of $\delta^{(4)}(k)$

Definition via inverse transform

Starting from

$$\delta^{(4)}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x},$$

we obtain, by symmetry of Fourier transform,

$$\boxed{\delta^{(4)}(k) = \int d^4x\, e^{ik\cdot x}}$$

Check

$$\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} \left(\int d^4y\, e^{ik\cdot y} \right) = \int d^4y\, \delta^{(4)}(x-y) = 1$$

Decomposition

$$\delta^{(4)}(k) = \delta(k^0)\, \delta^{(3)}(\mathrm{k})$$

$$\boxed{\delta^{(4)}(k) = \int d^4x\, e^{ik\cdot x}}$$