Quantisation - the complex scalar field

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The Complex Scalar Field

Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$

Equivalent to two real scalar fields:

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

But it is more convenient to treat:

$$\phi \quad \text{and} \quad \phi^\dagger$$

as independent fields.

Canonical Momenta

Define conjugate momenta:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger$$

$$\pi^\dagger(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi}$$

These are independent operators.

Canonical Quantization

Impose equal-time commutation relations:

$$[\phi(t,\vec{x}),\pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\phi^\dagger(t,\vec{x}),\pi^\dagger(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$

All other commutators vanish.

Mode Expansion

Field expansion:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_{\vec{k}} e^{-ikx} + b^\dagger_{\vec{k}} e^{ikx} \right)$$

$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a^\dagger_{\vec{k}} e^{ikx} + b_{\vec{k}} e^{-ikx} \right)$$

$$\omega_k = \sqrt{\vec{k}^2 + m^2}$$

Interpretation of Operators

- $a_{\vec{k}}$ annihilates particles
- $a^\dagger_{\vec{k}}$ creates particles
- $b_{\vec{k}}$ annihilates antiparticles
- $b^\dagger_{\vec{k}}$ creates antiparticles

Two types of excitations:

$$a^\dagger_{\vec{k}}|0\rangle, \quad b^\dagger_{\vec{k}}|0\rangle$$

Both with energy:

$$\omega_k = \sqrt{\vec{k}^2 + m^2}$$

Commutation Relations

$$[a_{\vec{k}}, a^\dagger_{\vec{p}}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})$$

$$[b_{\vec{k}}, b^\dagger_{\vec{p}}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})$$

All other commutators vanish.

Four-Momentum Operator

The conserved 4-momentum:

$$P^\mu = \int d^3k \, k^\mu \left(a^\dagger_{\vec{k}} a_{\vec{k}} + b^\dagger_{\vec{k}} b_{\vec{k}} \right)$$

In particular:

$$H = P^0 = \int d^3k \, \omega_k \left(a^\dagger_{\vec{k}} a_{\vec{k}} + b^\dagger_{\vec{k}} b_{\vec{k}} \right)$$

Global $U(1)$ Symmetry

The Lagrangian is invariant under:

$$\phi \to e^{i\alpha}\phi, \quad \phi^\dagger \to e^{-i\alpha}\phi^\dagger$$

Noether current:

$$j^\mu = -i\left(\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi\right)$$

Charge Operator

Define:

$$Q = \int d^3x\, j^0$$

$$Q = -i \int d^3x \left(\phi^\dagger \dot{\phi} - \dot{\phi}^\dagger \phi \right)$$

In terms of operators:

$$Q = \int \frac{d^3k}{(2\pi)^3} \left(a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}} \right)$$

Charge Eigenstates

$$Q\, a_{\vec{k}}^{\dagger}|0\rangle = +a_{\vec{k}}^{\dagger}|0\rangle$$

$$Q\, b_{\vec{k}}^{\dagger}|0\rangle = -b_{\vec{k}}^{\dagger}|0\rangle$$

Interpretation:

$$Q = N_{\text{particles}} - N_{\text{antiparticles}}$$

Summary

- Complex scalar field $\Rightarrow$ particles + antiparticles
- Quantization introduces a and b operators
- 4-momentum counts total quanta
- $U(1)$ symmetry $\Rightarrow$ conserved charge
- Charge distinguishes particles from antiparticles