Spin 1 - quantisation

Rui Santos

FCUL & CFTC

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Physical Fields

- The physical fields are $\vec{E}$ and $\vec{B}$
- Quantum theory is formulated in terms of the 4-potential A_μ

- A_μ has 4 degrees of freedom
- Photon has 2 polarisation states

Non-dynamical Component

- A_0 has no kinetic term $\dot{A}_0^2 \Rightarrow$ not dynamical
- It is determined by the equation of motion:

$$\nabla \cdot \vec{E} = 0$$

- Leads to a constraint equation:

$$\nabla^2 A_0 + \nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0$$

Gauge Symmetry

- The Lagrangian has a large symmetry group:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$$

- $\lambda(x)$ is arbitrary (vanishing at infinity)

- Field strength is invariant:

$$F_{\mu\nu} \to F_{\mu\nu}$$

Gauge Redundancy

- Two configurations related by a gauge transformation represent the same physical state

- Maxwell equations do not uniquely determine A_μ

What is the Problem? No conjugate momenta.

Canonical momenta:

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i$$

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

- $\Pi^0 = 0$ is a constraint
- Not all components are independent

What is the Problem? Equation not invertible.

$$\left[g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right] A^\nu = 0$$

- Not invertible
- Annihilates $\partial_\mu\lambda$

Resolution

- Identify gauge-equivalent configurations:

$$A_\mu \sim A_\mu + \partial_\mu\lambda$$

- They correspond to the same physical state

Quantisation of the Electromagnetic Field (Lorenz Gauge)

Gauge-fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Equations of motion

$$\Box A_\mu - \left(1 - \frac{1}{\alpha}\right) \partial_\mu (\partial \cdot A) = 0$$

Taking the divergence:

$$\Box (\partial \cdot A) = 0$$

$\Rightarrow \quad \partial \cdot A$ behaves as a massless scalar field

Gauge Fixing

- Different representatives of a physical state are related by gauge transformations
- Picking a gauge is choosing a representative

Lorenz gauge:

$$\partial_\mu A^\mu = 0$$

- Lorentz invariant
- Still allows residual gauge freedom

Coulomb Gauge

$$\nabla \cdot \vec{A} = 0$$

- Not Lorentz invariant
- Eliminates longitudinal modes
- Together with equations of motion $\Rightarrow A_0 = 0$ (free theory)

- Leaves only 2 transverse degrees of freedom

Summary

- A_μ has redundant degrees of freedom
- Gauge symmetry removes unphysical modes
- Only 2 transverse photon polarisation remain
- Constraints must be handled carefully in quantisation

Canonical Conjugate Momentum in General Covariant Gauge

Adding the Maxwell and gauge-fixing contributions,

$$\Pi^\mu = \Pi^\mu_{\text{Maxwell}} + \Pi^\mu_{\text{gf}}$$

with

$$\Pi^\mu_{\text{Maxwell}} = -F^{0\mu}$$

and

$$\Pi^\mu_{\text{gf}} = -\frac{1}{\alpha}\, g^{0\mu}\,(\partial_\nu A^\nu)\,,$$

we obtain the total canonical conjugate momentum:

$$\boxed{\Pi^\mu = -F^{0\mu} - \frac{1}{\alpha}\, g^{0\mu}\,(\partial_\nu A^\nu)}$$

This is the general result for the electromagnetic field in a covariant gauge with gauge parameter α.

Components of the Canonical Conjugate Momentum

Time component: $\mu = 0$

$$F^{00} = 0, \qquad g^{00} = 1,$$

we obtain

$$\Pi^0 = -\frac{1}{\alpha}(\partial_\nu A^\nu).$$

Thus, unlike pure Maxwell theory,

$$\Pi^0 \neq 0.$$

Spatial components: $\mu = i$

$$g^{0i} = 0,$$

the gauge-fixing term does not contribute, and

$$\Pi^i = -F^{0i}.$$

Using $F^{0i} = -E^i$ (with our sign convention), $\Pi^i = E^i$.

Quantisation of the Electromagnetic Field

We start from the mode expansion

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(\mathrm{k}) \left(a_{\mathrm{k}\lambda} e^{-ik\cdot x} + a_{\mathrm{k}\lambda}^\dagger e^{ik\cdot x} \right)$$

with $\omega_{\mathrm{k}} = |\mathrm{k}|$.

Polarisation completeness (covariant basis):

$$\epsilon_\mu^{(\lambda)} \epsilon^{(\lambda')\mu} = g^{\lambda\lambda'}$$

Conjugate Momentum Field

From the gauge-fixed Lagrangian, the canonical momentum is

$$\Pi^{\mu} = -F^{0\mu} - \frac{1}{\alpha} g^{0\mu}(\partial_{\nu} A^{\nu})$$

In terms of electric field (spatial components):

$$\Pi^{i} = F^{i0} = E^{i}$$

and in covariant form we expand:

$$\Pi^{\mu}(x) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathrm{k}}}{2}} \sum_{\lambda=0}^{3} \epsilon^{(\lambda)\mu}(\mathrm{k}) \left(a_{\mathrm{k}\lambda} e^{-ik\cdot x} - a^{\dagger}_{\mathrm{k}\lambda} e^{ik\cdot x} \right)$$

Canonical Commutation Relations

Equal-time quantisation imposes

$$[A_\mu(\mathrm{x}, t), \Pi^\nu(\mathrm{y}, t)] = i\delta_\mu{}^\nu \delta^{(3)}(\mathrm{x} - \mathrm{y})$$

All other commutators vanish:

$$[A_\mu, A_\nu] = 0, \qquad [\Pi^\mu, \Pi^\nu] = 0$$

Extracting $a_{\mathrm{k}\lambda}$

We invert the mode expansion using orthogonality:

$$a_{\mathrm{k}\lambda} = \int d^3x \, e^{ik\cdot x} \left[\sqrt{\frac{\omega_\mathrm{k}}{2}} \epsilon_\mu^{(\lambda)} A^\mu(x) + \frac{i}{\sqrt{2\omega_\mathrm{k}}} \epsilon_\mu^{(\lambda)} \Pi^\mu(x) \right]$$

and similarly for $a^\dagger$.

This step uses:

$$\int d^3x \, e^{i(\mathrm{k}-\mathrm{k}')\cdot \mathrm{x}} = (2\pi)^3 \delta^{(3)}(\mathrm{k}-\mathrm{k}')$$

Commutation Relations of $a, a^\dagger$

Using the canonical algebra of A_μ and Π^μ, we obtain

$$\left[a_{\mathrm{k}\lambda}, a^\dagger_{\mathrm{k}'\lambda'}\right] = -\,g_{\lambda\lambda'}(2\pi)^3\delta^{(3)}(\mathrm{k}-\mathrm{k}')$$

All other commutators vanish:

$$[a_{\mathrm{k}\lambda}, a_{\mathrm{k}'\lambda'}] = 0, \qquad [a^\dagger_{\mathrm{k}\lambda}, a^\dagger_{\mathrm{k}'\lambda'}] = 0$$

Mode Expansion and Polarisation in Lorenz Gauge

We introduce four polarisation vectors

$$\varepsilon_\mu^{(\lambda)}(k), \qquad \lambda = 0, 1, 2, 3$$

with completeness relation

$$\sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)}(k) = g_{\mu\nu}.$$

Choose momentum along z:

$$k^\mu = (k^0, 0, 0, k^3).$$

A convenient basis is

$$\varepsilon^{(0)} = (1, 0, 0, 0) \quad \text{(time-like)}$$

$$\varepsilon^{(1)} = (0, 1, 0, 0), \quad \varepsilon^{(2)} = (0, 0, 1, 0) \quad \text{(transverse)}$$

$$\varepsilon^{(3)} = (0, 0, 0, 1) \quad \text{(longitudinal)}$$

Field expansion

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathrm{k}}} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(k) \left(a_\mathrm{k}^{(\lambda)} e^{-ikx} + a_\mathrm{k}^{(\lambda)\dagger} e^{ikx} \right)$$

Commutation relations

$$[a_\mathrm{k}^{(\lambda)}, a_{\mathrm{k}'}^{(\lambda')\dagger}] = -\, g^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\mathrm{k} - \mathrm{k}')$$

$$[a, a] = [a^\dagger, a^\dagger] = 0$$

Important consequence

For the time-like mode:

$$\langle 0 | a_\mathrm{k}^{(0)} a_\mathrm{k}^{(0)\dagger} | 0 \rangle = -(2\pi)^3 \delta^{(3)}(\mathrm{k} - \mathrm{k}')$$

$\Rightarrow$ **negative norm states**

How do we solve this?.

Gupta–Bleuler Decomposition

In covariant quantisation the gauge field is split into positive- and negative-frequency parts:

$$A_\mu(x) = A_\mu^{(+)}(x) + A_\mu^{(-)}(x)$$

Positive-frequency part (annihilation operators):

$$A_\mu^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(\mathrm{k})\, a_{\mathrm{k}\lambda}\, e^{-ik\cdot x}$$

Negative-frequency part (creation operators):

$$A_\mu^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(\mathrm{k})\, a_{\mathrm{k}\lambda}^\dagger\, e^{ik\cdot x}$$

Gupta–Bleuler Condition and Physical States

We must eliminate the unphysical polarisation (time-like and longitudinal).

Gupta–Bleuler condition
Instead of imposing $\partial_\mu A^\mu = 0$ as an operator identity (wrong commutation relations), we require it only on physical states:

$$\partial_\mu A^{\mu(+)}(x)\,|\text{phys}\rangle = 0$$

where $A^{(+)}$ is the positive-frequency part.

Using the mode expansion:

$$\partial_\mu A^\mu(x) \sim \sum_{\lambda=0}^{3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \left(k\cdot\varepsilon^{(\lambda)}\right) a_{\mathrm{k}}^{(\lambda)} e^{-ikx} + \text{h.c.}$$

Gupta–Bleuler Condition in Momentum Space

We impose the physical state condition

$$\partial^\mu A_\mu^{(+)}(x)\,|\text{phys}\rangle = 0.$$

Insert the mode expansion:

$$A_\mu^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(\mathrm{k}) a_{\mathrm{k}\lambda} e^{-ik\cdot x}.$$

Then

$$\partial^\mu \to -ik^\mu,$$

so

$$\partial^\mu A_\mu^{(+)}(x) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathrm{k}}}{2}} \sum_{\lambda} \left(k \cdot \epsilon^{(\lambda)}\right) a_{\mathrm{k}\lambda} e^{-ik\cdot x}.$$

Special Frame: Lightlike Momentum

Choose the standard frame

$$k^\mu = (\omega, 0, 0, \omega), \qquad k^2 = 0.$$

We choose polarization vectors such that:

$$k \cdot \epsilon^{(1)} = 0, \qquad k \cdot \epsilon^{(2)} = 0$$

(transverse modes),

$$k \cdot \epsilon^{(0)} = \omega, \qquad k \cdot \epsilon^{(3)} = \omega$$

(time-like and longitudinal modes).

Thus only unphysical modes contribute to the divergence:

Reduction of the Gupta–Bleuler Condition

The condition

$$\partial^\mu A_\mu^{(+)}|\text{phys}\rangle = 0$$

becomes

$$\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\left[\omega\left(a_k^{(0)} - a_k^{(3)}\right)\right] e^{-ik\cdot x}|\text{phys}\rangle = 0.$$

Since plane waves are independent, we obtain:

$$\boxed{\left(a_k^{(0)} - a_k^{(3)}\right)|\text{phys}\rangle = 0}$$

This removes the combination of longitudinal and time-like modes.

Physical Hilbert Space Structure

A general one-particle state is

$$|\psi\rangle = \sum_{\lambda=0}^{3} \int d^3k \, c_\lambda(\mathrm{k}) a^\dagger_{\mathrm{k}\lambda} |0\rangle.$$

The Gupta–Bleuler condition implies:

$$c_0(\mathrm{k}) = c_3(\mathrm{k})$$

so only the combination

$$a^{(0)} - a^{(3)}$$

is constrained.

Physical states:

$$|\text{phys}\rangle = \text{states generated only by } a^{(1)\dagger}, a^{(2)\dagger}.$$

2 transverse photon polarizations remain

Hamiltonian in Covariant Quantisation

The Hamiltonian reads

$$H = \int d^3k\, \omega_k \left(\sum_{\lambda=1}^{2} a_k^{(\lambda)\dagger} a_k^{(\lambda)} - a_k^{(0)\dagger} a_k^{(0)} + a_k^{(3)\dagger} a_k^{(3)} \right).$$

Using

$$(a^{(0)} - a^{(3)})|\text{phys}\rangle = 0,$$

all unphysical contributions cancel in expectation values:

$$\langle \text{phys}|H|\text{phys}\rangle = \langle \text{phys}|H_T|\text{phys}\rangle.$$

Only transverse photons contribute physically

EXTRA SLIDES ON GAUGE FIXING

Covariant Gauge Fixing and Landau Gauge

For the electromagnetic field, we add the gauge-fixing term

$$\mathcal{L}_{\rm gf} = -\frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2$$

to the Maxwell Lagrangian.
Thus,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2$$

where:

- $\alpha = 1$: Feynman gauge
- $\alpha \to 0$: Landau gauge

The Landau gauge is obtained as the limit $\alpha \to 0$, not by directly substituting $\alpha = 0$.

Why Does $\alpha \to 0$ Impose $\partial \cdot A = 0$?

The path integral is

$$Z = \int \mathcal{D}A \; \exp\left[i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial \cdot A)^2\right)\right].$$

The gauge-fixing factor contributes

$$\exp\left[-\frac{i}{2\alpha}\int d^4x\,(\partial \cdot A)^2\right].$$

If

$$\partial_\mu A^\mu \neq 0,$$

then for $\alpha \to 0$

$$\frac{1}{\alpha} \to \infty$$

and the configuration becomes infinitely suppressed.

Landau Gauge as a Strict Constraint

Only field configurations satisfying

$$\partial_\mu A^\mu = 0$$

remain unsuppressed in the limit

$$\alpha \to 0.$$

Therefore:

$$\boxed{\text{Landau gauge} \;=\; \partial_\mu A^\mu = 0}$$

This is the Lorenz condition imposed exactly.

Interpretation:

- finite α : soft enforcement
- small α : strong preference
- $\alpha \to 0$: exact constraint

This is why Landau gauge is the sharp limit of covariant gauges.

Why Not Simply Set $\alpha = 0$?

Direct substitution gives

$$-\frac{1}{2\alpha}(\partial \cdot A)^2$$

which is singular at

$$\alpha = 0.$$

So Landau gauge must be understood as

$$\boxed{\alpha \to 0}$$

rather than literal substitution.

A cleaner and fully regular formulation uses an **Nakanishi–Lautrup auxiliary field**.

Auxiliary Field Formulation

Introduce an auxiliary scalar field $B(x)$:

$$\mathcal{L}_{\rm gf} = B\,\partial_\mu A^\mu + \frac{\alpha}{2} B^2$$

where:

- B has no kinetic term
- B has no propagating degrees of freedom
- it is purely auxiliary

This formulation is exactly equivalent to

$$-\frac{1}{2\alpha}(\partial \cdot A)^2$$

for $\alpha \neq 0$.

Equation of Motion for the Auxiliary Field

Varying with respect to B gives

$$\frac{\partial \mathcal{L}}{\partial B} = \partial_\mu A^\mu + \alpha B = 0$$

therefore

$$B = -\frac{1}{\alpha}\partial_\mu A^\mu$$

Substituting back:

$$\mathcal{L}_{\mathrm{gf}} = B(\partial \cdot A) + \frac{\alpha}{2}B^2$$

gives

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\alpha}(\partial \cdot A)^2$$

which reproduces the usual gauge-fixing term.

Landau Gauge from the Auxiliary Field

Now we can safely set

$$\alpha = 0$$

in the auxiliary-field form:

$$\mathcal{L}_{\rm gf} = B\,\partial_\mu A^\mu$$

with no singularity.

The equation of motion for B becomes

$$\boxed{\partial_\mu A^\mu = 0}$$

exactly.
Thus B acts as a **Lagrange multiplier** enforcing the Landau gauge condition.

Path Integral Interpretation

In the path integral:

$$\int \mathcal{D}B \, \exp\left[i \int d^4x \, B(\partial \cdot A) \right]$$

integration over B produces

$$\delta(\partial_\mu A^\mu)$$

therefore

$$\boxed{\text{only fields with } \partial_\mu A^\mu = 0 \text{ contribute}}$$

This shows explicitly that Landau gauge is an exact constraint on the functional integral.

Final Summary

Two equivalent ways to understand Landau gauge:

Method 1: Limit of covariant gauges

$$\alpha \to 0$$

in

$$-\frac{1}{2\alpha}(\partial \cdot A)^2$$

which suppresses all configurations with

$$\partial_\mu A^\mu \neq 0$$

Method 2: Auxiliary field approach

$$\mathcal{L}_{\rm gf} = B\,\partial_\mu A^\mu + \frac{\alpha}{2}B^2$$

and for $\alpha = 0$, B imposes