Direct Detection: Effective Lagrangian, Quark & Gluon Contributions

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Dark Matter in the Galaxy

- We do not directly observe dark matter (DM)
- Only indirect measurements over large scales

We need to determine the local phase-space distribution $f(\vec{x}, \vec{v})$

Relevant for:

- Direct detection experiments
- Event rate predictions

Standard Halo Model (SHM)

Assumption for DM distributions

$$\rho(r) \sim r^{-2}$$

Velocity distribution:

$$f(v) \propto e^{-\frac{mv^2}{2kT}}$$

Typical values:

- $v_0 \sim 220$ km/s (from the observed rotation speed of the Milky Way at the Sun's position)
- $v_{esc} \sim 530$ km/s

Important:

$$f(v) = 0 \quad \text{for } v > v_{esc}$$

Maxwell-Boltzmann Distribution

From statistical mechanics:

$$f(v)\, d^3v = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}\, d^3v$$

Assumptions:

- Non-relativistic particles
- Thermal equilibrium
- No interactions

Why Velocity Distribution Matters

$$\text{Detection rate} = \text{particle physics} \times \text{astrophysics}$$

Factorization:

- Cross section $\rightarrow$ particle physics
- $f(v) \rightarrow$ astrophysics

The rate is defined by

$$R \propto \int f(v)\, v\, \sigma\, d^3v$$

Why is $f(\text{v})v\sigma d^3v$ a rate?

Velocity distribution:

$$f(\text{v})d^3v = \text{number density of particles with velocity v}$$

Flux of particles:

$$\text{flux} = (\text{density}) \times (\text{speed}) = f(\text{v})vd^3v$$

- Flux = number of particles crossing unit area per unit time
- The factor v converts density into a flow

Including interactions:

$$d\Gamma = (\text{flux}) \times \sigma = f(\text{v})v\sigma d^3v$$

Physical Interpretation

Total interaction rate:

$$\Gamma = \int f(\mathrm{v}) v \sigma d^3 v$$

Interpretation:

- $f(\mathrm{v}) d^3 v$ = number density
- v = speed converts density into flux
- σ = interaction area

$$\text{rate} = \text{density} \times \text{speed} \times \text{cross section}$$

Direct Detection: Physical Picture

- Earth moves through DM halo
- DM scatters off nuclei
- Energy deposited: recoil energy E_R

Typical scales:

- $v/c \sim 10^{-3}$
- $E_R \sim \text{keV}$

Differential Event Rate

$$\frac{dR}{dE_R} = N_T \frac{\rho_0}{m_\chi} \int_{v > v_{\min}} d^3v \, f(\vec{v} + \vec{v}_E(t)) \, v \frac{d\sigma}{dE_R}$$

Ingredients:

- ρ_0: local DM density
- m_χ: DM mass
- N_T: number of targets
- v_E: Earth's velocity

Minimum Velocity in Dark Matter Scattering

Setup: Dark matter mass: m_χ; Nucleus mass: m_N; Recoil energy: E_R; Reduced mass:

$$\mu = \frac{m_\chi m_N}{m_\chi + m_N}$$

Recoil energy (elastic scattering):

$$E_R = \frac{\mu^2}{m_N} v^2 (1 - \cos\theta)$$

Maximum recoil ($\theta = \pi$):

$$E_R^{\max} = \frac{2\mu^2}{m_N} v^2$$

Minimum velocity for a given E_R:

$$\boxed{v_{\min} = \sqrt{\frac{m_N E_R}{2\mu^2}}}$$

Differential Cross Section

Interpretation:

- Large E_R requires large v
- Only fast DM particles contribute at high recoil
- High E_R probes the high-velocity tail of the distribution

$$\frac{d\sigma}{dE_R} = \frac{m_N}{2\mu^2 v^2}\left(\sigma_{\mathrm{SI}} F_{\mathrm{SI}}^2(E_R) + \sigma_{\mathrm{SD}} F_{\mathrm{SD}}^2(E_R)\right)$$

Two contributions:

- Spin-independent (SI)
- Spin-dependent (SD)

Nuclear Form Factors: Finite Size Effects

Note: the nucleus is an extended object, not point-like

- Dark matter scatters off a nucleus at rest
- Momentum transfer: $\vec{q} = \vec{p}_N^{\text{final}}$

Non-relativistic recoil:

$$E_R = \frac{p_N^2}{2m_N}$$

Solve for momentum:

$$p_N = \sqrt{2m_N E_R}$$

Therefore:

$$\boxed{q = \sqrt{2m_N E_R}}$$

When $qR \ll 1$:

$$F(q) \approx 1 \quad \text{(coherent scattering)}$$

When $qR \gtrsim 1$:

$$F(q) < 1 \quad \text{(loss of coherence)}$$

Physical picture:

- Small $q \to$ DM sees whole nucleus $\Rightarrow$ amplitudes add coherently
- Large $q \to$ resolves nuclear structure $\Rightarrow$ destructive interference

Effect on rate:

$$\frac{dR}{dE_R} \propto |F(q)|^2$$

- Suppresses high-recoil (E_R) events
- Encodes nuclear structure

Spin-Independent (SI) Interactions

Coherent scattering off the nucleus:

$$\mathcal{A} \sim \sum_{i=1}^{A} e^{i\vec{q}\cdot\vec{x}_i}$$

If $qR \ll 1$:

$$e^{i\vec{q}\cdot\vec{x}_i} \approx 1 \quad \Rightarrow \quad \mathcal{A} \sim A$$

$$\Rightarrow \quad \sigma \propto |\mathcal{A}|^2 \sim A^2$$

- Dark matter couples to **mass density**
- All nucleons contribute coherently

Spin-Independent Cross Section

$$\sigma_{\mathrm{SI}} \propto \mu^2 \left|Z g_p + (A - Z) g_n\right|^2$$

Special case: $g_p \simeq g_n$

$$\sigma_{\mathrm{SI}} \propto \mu^2 A^2$$

Including form factor:

$$\frac{d\sigma}{dE_R} \propto A^2 \left|F(q)\right|^2$$

Implication:

- Heavy nuclei (Xe, Ge) strongly enhance signal
- Suppression only at large q via $F(q)$

Spin-Dependent (SD) Interactions

Couples to nuclear spin:

$$\mathcal{A} \sim \langle \vec{S}_p \rangle g_p + \langle \vec{S}_n \rangle g_n$$

Cross section:

$$\sigma_{\mathrm{SD}} \propto \mu^2 \left(g_p \langle S_p \rangle + g_n \langle S_n \rangle \right)^2$$

- No coherent sum over all nucleons
- Only **unpaired nucleons** contribute
- Depends on nuclear spin structure

SI vs SD: Physical Picture

Spin-Independent (SI):

- Couples to mass density
- All nucleons add coherently
- $\sigma \propto A^2$ (large enhancement)

Spin-Dependent (SD):

- Couples to nuclear spin
- Only unpaired nucleons contribute
- No A^2 enhancement

Experimental consequence:

- SI dominates for heavy targets (Xe, Ge)
- SD requires nuclei with non-zero spin (e.g. ^{19}F, ^{129}Xe)

Link to Particle Physics

Microscopic theory lives at quark level:

$$\mathcal{L}_{\chi q} \sim (\bar{\chi}\Gamma\chi)(\bar{q}\Gamma q)$$

But experiments measure:

$$\text{DM} + \text{nucleus} \to \text{nucleus recoil}$$

So we need a hierarchy of scales:

- DM-quark interaction (fundamental theory)
- $\Downarrow$ QCD confinement
- DM-nucleon interaction (hadronic physics)
- $\Downarrow$ nuclear binding
- DM-nucleus interaction (experiment)

From Quarks to Nuclei: Matching Problem

Step 1: quark-level operator

$$(\bar{\chi}\Gamma\chi)(\bar{q}\Gamma q)$$

Step 2: hadronization

$$\langle N|\bar{q}\Gamma q|N\rangle \quad \Rightarrow \quad \text{nucleon-level couplings}$$

Step 3: nuclear embedding

$$\langle A|\sum_{i=1}^{A}\mathcal{O}_i|A\rangle \quad \Rightarrow \quad \text{coherence + form factors}$$

- Each step integrates out shorter-distance physics
- New effective degrees of freedom appear at each scale

Effective Field Theory (EFT)

Idea: integrate out heavy mediator

$$\mathcal{L}_{\text{full}} \longrightarrow \mathcal{L}_{\text{eff}}$$

Generic low-energy operator:

$$\mathcal{L}_{\text{eff}} = \sum_i \frac{C_i}{\Lambda^2} (\bar{\chi}\Gamma_i \chi)(\bar{q}\Gamma_i q)$$

Interpretation:

- Λ = mediator mass scale
- C_i = model-dependent couplings
- Valid for $q \ll \Lambda$

Matching to a UV Theory

Example: heavy vector mediator W'-like exchange

$$\mathcal{L} \sim \frac{g_\chi g_q}{q^2 - M^2}(\bar{\chi}\gamma^\mu\chi)(\bar{q}\gamma_\mu q)$$

Low-energy limit $q^2 \ll M^2$:

$$\frac{1}{q^2 - M^2} \simeq -\frac{1}{M^2}$$

$$\Rightarrow \mathcal{L}_{\text{eff}} = -\frac{g_\chi g_q}{M^2}(\bar{\chi}\gamma^\mu\chi)(\bar{q}\gamma_\mu q)$$

Fermi Theory Analogy

Standard Model example:

$$\mathcal{L} \sim \frac{g^2}{8} \frac{1}{q^2 - m_W^2} J^\mu J_\mu$$

For $q^2 \ll m_W^2$:

$$\mathcal{L}_{\text{eff}} = \frac{g^2}{8 m_W^2} J^\mu J_\mu$$

$$\Rightarrow G_F = \frac{g^2}{4\sqrt{2} m_W^2}$$

Lesson: heavy mediators become contact interactions

Why EFT is Powerful

Separation of scales:

- UV physics: unknown heavy particles
- IR physics: measurable scattering

Advantages:

- Model-independent parameterization
- Systematic expansion in q/Λ
- Universal comparison across experiments

Full Picture of Direct Detection

Three ingredients:

- Astrophysics:

$$f(v) \quad \text{(velocity distribution)}$$

- Particle physics:

$$\sigma(q) \quad \text{(EFT operators)}$$

- Nuclear physics:

$$F(q),\ \langle S_{p,n} \rangle$$

Event rate factorization:

$$\frac{dR}{dE_R} \sim f(v) \otimes \sigma(q) \otimes |F(q)|^2$$

Summary

- Direct detection connects quark-level physics to nuclei
- EFT encodes heavy physics in contact operators
- Nuclear physics enters via form factors and spin structure
- Observables factorize into astrophysics, particle physics, and nuclear physics

Physics goal and kinematic regime

- Goal: spin-independent (SI) elastic scattering of scalar-coupled DM off nucleons.
- Direct detection momentum transfer is tiny: $q^2 \simeq -|\boldsymbol{q}|^2 \ll m_{h_i}^2$.
- Strategy: integrate out heavy mediators $\Rightarrow$ EFT operators for quarks/gluons.

Approximation

For $|q^2| \ll m_{h_i}^2$,

$$\frac{1}{q^2 - m_{h_i}^2} \approx -\frac{1}{m_{h_i}^2} \quad \text{(up to } \mathcal{O}(q^2/m_{h_i}^2)\text{)}.$$

Effective Lagrangian at leading order (operator basis)

We write (scalar and twist-2 contributions)

$$\mathcal{L}_{\text{eff}} = \sum_q C_q^S \, \mathcal{O}_q^S + C_g^S \, \mathcal{O}_g^S + \sum_q C_q^T \, \mathcal{O}_q^T \, .$$

Parton-level operators:

$$\mathcal{O}_q^S = m_q \, \bar{q} q, \qquad \mathcal{O}_g^S = \frac{\alpha_s}{\pi} \, G_{\mu\nu}^a G^{a\mu\nu},$$

$$\mathcal{O}_q^T = \frac{1}{m_\chi^2} \, \chi \, i\partial^\mu i\partial^\nu \chi \, \mathcal{O}_{\mu\nu}^q .$$

Comments

- $\mathcal{O}_q^S, \mathcal{O}_g^S$ drive SI scattering in the zero-momentum limit.
- Twist-2 encodes coupling to the QCD energy-momentum tensor (parton momentum fractions).

Tree-level quark contribution from scalar mediation

Generic t-channel amplitude (sum over scalar mediators h_i):

$$\mathcal{A}_{\text{gen}} = \sum_i C_{\chi\chi h_i}\, C_{qqh_i}\, \frac{1}{q^2 - m_{h_i}^2}\, \bar{u}(p)\, u(p+q).$$

Small momentum transfer $q^2 \to 0$:

$$\mathcal{A}_{\text{gen}} \xrightarrow[q^2\to 0]{} -\sum_i C_{\chi\chi h_i}\, C_{qqh_i}\, \frac{1}{m_{h_i}^2}\, \bar{u}(p)\, u(p).$$

Matching idea

Match this onto EFT term $\sim \chi\chi\, \bar{q}q$ and read off C_q^S.

From amplitude to EFT: extracting C_q^S

Assuming scalar-like couplings, the EFT contains

$$\mathcal{L}_{\text{eff}} \supset -\sum_i \frac{C_{\chi\chi h_i}\, C_{qqh_i}}{2m_{h_i}^2}\, \chi\chi\, \bar{q}q.$$

Using $\mathcal{O}_q^S = m_q \bar{q}q$:

$$\mathcal{L}_{\text{eff}} \supset \sum_q C_q^S\, m_q\, \bar{q}q,$$

so the Wilson coefficient is

$$\boxed{C_q^S \;=\; -\sum_i \frac{C_{\chi\chi h_i}\, C_{qqh_i}}{2\, m_q\, m_{h_i}^2}}\,.$$

Important

At tree level, t-channel scalar exchange is the only topology for this operator in many scalar-mediator models, but loop/topology variations can add extra pieces to C_q^S.

Nucleon matrix elements: scalar operators

To go from partons to nucleons ($N = p, n$), use

$$\langle N | m_q \bar{q} q | N \rangle = m_N \, f_{Tq}^N ,$$

and define

$$f_{TG}^N \equiv 1 - \sum_{q=u,d,s} f_{Tq}^N .$$

For the gluon operator,

$$\frac{9\alpha_s}{8\pi} \langle N | G^a_{\mu\nu} G^{a\mu\nu} | N \rangle = m_N \, f_{TG}^N .$$

Interpretation

Light-quark scalar content is encoded in f_{Tq}^N; the remainder of the nucleon mass is largely gluonic (f_{TG}^N).

Twist-2 nucleon matrix element (structure function moments)

For the twist-2 operator, the nucleon matrix element is parameterised by

$$\langle N(p)|\mathcal{O}^q_{\mu\nu}|N(p)\rangle = \frac{1}{m_N}\left(p_\mu p_\nu - \frac{1}{4}m_N^2 g_{\mu\nu}\right)\left[q^N(2) + \bar{q}^N(2)\right],$$

where $q^N(2)$ and $\bar{q}^N(2)$ are the second moments of PDFs.

What enters the SI coupling

Only the combination $q^N(2) + \bar{q}^N(2)$ matters at this order.

Putting it together: effective DM–nucleon coupling f_N

Taking the nucleon expectation value of $\mathcal{L}_{\text{eff}}$ and using the standard matrix elements yields

$$\boxed{\frac{f_N}{m_N} = \sum_{q=u,d,s} C_S^q\, f_{Tq}^N \;+\; \frac{3}{4} \sum_{q=u,d,s,c,b} \left[q^N(2) + \bar{q}^N(2)\right] C_T^q \;-\; \frac{8\pi}{9\alpha_s}\, f_{TG}^N\, C_S^g}$$

with

$$f_{TG}^N \equiv 1 - \sum_{q=u,d,s} f_{Tq}^N.$$

- First term: scalar couplings to light quarks (u, d, s).
- Second term: twist-2 (include all quarks below $\sim$1 GeV; in practice all but top).
- Third term: scalar gluon operator in our basis $\mathcal{O}_g^S = (\alpha_s/\pi)G^2$.

SI cross section (per nucleon)

Define the reduced mass

$$\mu_{\chi N} = \frac{m_\chi m_N}{m_\chi + m_N}.$$

The spin-independent cross section is

$$\boxed{\sigma_N^{\mathrm{SI}} = \frac{\mu_{\chi N}^2}{\pi} |f_N|^2}$$

Higgs portal (real scalar DM): setup and matching

Portal interaction (real scalar χ):

$$\mathcal{L} \supset -\lambda_p \, \chi^2 \, H^\dagger H.$$

After EWSB, $H \to \frac{1}{\sqrt{2}}(0, v + h)$,

$$\mathcal{L} \supset -\lambda_p \, v \, \chi^2 h \; + \; \cdots \qquad \Rightarrow \qquad C_{\chi\chi h} = 2\lambda_p v.$$

SM Yukawa interaction:

$$\mathcal{L} \supset -\sum_q \frac{m_q}{v} \, h \, \bar{q} q \qquad \Rightarrow \qquad C_{qqh} = \frac{m_q}{v}.$$

Tree-level matching at $q^2 \to 0$:

$$\mathcal{L}_{\text{eff}} \supset -\frac{C_{\chi\chi h}\, C_{qqh}}{2m_h^2}\, \chi\chi\, \bar{q}q = -\frac{2\lambda_p v}{2m_h^2}\frac{m_q}{v}\, \chi^2\, \bar{q}q.$$

Using $\mathcal{O}_q^S = m_q \bar{q}q$, we identify

$$\boxed{C_S^q = -\frac{\lambda_p}{m_h^2} \qquad (q = u, d, s, c, b, t)}$$

Higgs portal: f_N at LO (scalar operators only)

Neglecting twist-2 at LO and integrating out heavy quarks in the standard way gives

$$\boxed{\frac{f_N}{m_N} = C_S^q \left(\sum_{q=u,d,s} f_{Tq}^N + \frac{2}{27} f_{TG}^N \sum_{Q=c,b,t} 1 \right) = C_S^q \left(\sum_{q=u,d,s} f_{Tq}^N + \frac{2}{9} f_{TG}^N \right)}$$

so that

$$\boxed{f_N = -\frac{\lambda_p}{m_h^2}\, m_N \left(\sum_{q=u,d,s} f_{Tq}^N + \frac{2}{9} f_{TG}^N \right)}$$

Often one defines the dimensionless hadronic factor

$$f^{(N)} \equiv \sum_{q=u,d,s} f_{Tq}^N + \frac{2}{9} f_{TG}^N \approx 0.30 \quad \Rightarrow \quad f_N = -\frac{\lambda_p}{m_h^2}\, m_N\, f^{(N)}.$$

Higgs portal: SI cross section

With $\sigma_N^{\rm SI} = \mu_{\chi N}^2 |f_N|^2/\pi$ and $f_N = -(\lambda_p/m_h^2) m_N f^{(N)}$,

$$\boxed{\sigma_N^{\rm SI} = \frac{\mu_{\chi N}^2}{\pi} \left(\frac{\lambda_p\, m_N}{m_h^2}\right)^2 \left(f^{(N)}\right)^2}$$

- Heavy DM limit: $\mu_{\chi N} \to m_N$, so $\sigma_N^{\rm SI}$ approaches a constant.
- Event rate still falls as $1/m_\chi$ because the DM number density is ρ_χ/m_χ.
- Flavor-universal coefficient

Comparison with LZ limits (how to constrain λ_p)

Using

$$\sigma_N^{\rm SI} = \frac{\mu_{\chi N}^2}{\pi} \left(\frac{\lambda_p \, m_N}{m_h^2} \right)^2 \left(f^{(N)} \right)^2, \qquad \mu_{\chi N} = \frac{m_\chi m_N}{m_\chi + m_N},$$

the LZ upper limit $\sigma_N^{\rm lim}(m_\chi)$ implies

$$\boxed{\lambda_p \leq \frac{m_h^2}{m_N f^{(N)}} \sqrt{\frac{\pi \, \sigma_N^{\rm lim}(m_\chi)}{\mu_{\chi N}^2}}}$$

- The strongest LZ SI limits occur around $m_\chi \sim \mathcal{O}(30\text{–}40)\,\text{GeV}$.
- In the heavy-DM limit $\mu_{\chi N} \to m_N$, so the bound on λ_p becomes mass-independent.

Benchmarks: translating LZ-like bounds into λ_p

Assume Higgs portal real scalar DM with $m_h = 125$ GeV, $m_N = 0.939$ GeV, $f^{(N)} = 0.30$ and an illustrative bound $\sigma_N^{\rm lim} = 6 \times 10^{-48}\,\mathrm{cm}^2$.

$$\lambda_p = \frac{m_h^2}{m_N f^{(N)}} \sqrt{\frac{\pi\, \sigma_N^{\rm lim}}{\mu_{\chi N}^2}}, \qquad \mu_{\chi N} = \frac{m_\chi m_N}{m_\chi + m_N}.$$

Results

- $m_\chi = 50$ GeV: $\mu_{\chi N} \simeq 0.922$ GeV $\Rightarrow$ $\boxed{\lambda_p \lesssim 1.3 \times 10^{-5}}$
- $m_\chi = 200$ GeV: $\mu_{\chi N} \simeq 0.934$ GeV $\Rightarrow$ $\boxed{\lambda_p \lesssim 1.3 \times 10^{-5}}$

(Nearly identical because $\mu_{\chi N} \to m_N$ once $m_\chi \gg m_N$. Use $\sigma_N^{\rm lim}(m_\chi)$ from the experimental curve for a mass-accurate bound.)

Fermionic Higgs portal (EFT operator and Higgs couplings)

Gauge-invariant fermionic Higgs portal (dimension-5 operator):

$$\mathcal{L} \supset -\frac{y_\chi}{\Lambda}\,\bar{\chi}\chi\,H^\dagger H,$$

where χ is a (Dirac or Majorana) fermion and y_χ is dimensionless.
After EWSB, $H \to \frac{1}{\sqrt{2}}(0, v+h)$,

$$H^\dagger H = \frac{(v+h)^2}{2} = \frac{v^2}{2} + vh + \frac{h^2}{2},$$

so

$$\mathcal{L} \supset -\frac{y_\chi}{\Lambda}\,\bar{\chi}\chi\left(\frac{v^2}{2} + vh + \frac{h^2}{2}\right).$$

Couplings generated

$$g_{\chi\chi h} = \frac{y_\chi v}{\Lambda}, \qquad g_{\chi\chi hh} = \frac{y_\chi}{2\Lambda}.$$

The SM Yukawa coupling is $g_{hqq} = m_q/v$.

Tree-level amplitude and matching to C_S^q

t-channel Higgs exchange (small momentum transfer $q^2 \to 0$):

$$\mathcal{A} = \left(\frac{y_\chi v}{\Lambda}\right) \frac{1}{q^2 - m_h^2} \left(\frac{m_q}{v}\right) \bar{u}_\chi u_\chi \; \bar{u}_q u_q \xrightarrow[q^2 \to 0]{} -\frac{y_\chi}{\Lambda m_h^2} m_q \, \bar{u}_\chi u_\chi \; \bar{u}_q u_q .$$

Match onto the EFT basis

$$\mathcal{L}_{\text{eff}} \supset \sum_q C_S^q \, \mathcal{O}_S^q, \qquad \mathcal{O}_S^q = m_q \bar{q} q,$$

i.e. onto $(\bar{\chi}\chi)(m_q \bar{q} q)$:

$$\boxed{C_S^q = -\frac{y_\chi}{\Lambda m_h^2} \qquad (q = u, d, s, c, b, t)}$$

- Flavor-universal coefficient (the m_q is inside $\mathcal{O}_S^q$).
- Overall sign is irrelevant for σ at LO (it enters as $|f_N|^2$).

From quarks to nucleons: f_N for fermionic Higgs portal (LO)

Neglect twist-2 at LO. Integrating out heavy quarks gives

$$\boxed{\frac{f_N}{m_N} = C_S^q \left(\sum_{q=u,d,s} f_{Tq}^N + \frac{2}{9} f_{TG}^N \right), \qquad f_{TG}^N \equiv 1 - \sum_{q=u,d,s} f_{Tq}^N.}$$

Therefore,

$$\boxed{f_N = -\frac{y_\chi}{\Lambda m_h^2}\, m_N\, f^{(N)}, \qquad f^{(N)} \equiv \sum_{q=u,d,s} f_{Tq}^N + \frac{2}{9} f_{TG}^N \approx 0.30.}$$

Dimensional check

$[C_S^q] = M^{-3}$, hence $[f_N] = M^{-2}$ and $\sigma_N^{\mathrm{SI}} = \mu^2 |f_N|^2/\pi$ has $[\sigma] = M^{-2}$.

Fermionic DM SI cross section (per nucleon)

$$\sigma_N^{\mathrm{SI}} = \frac{\mu_{\chi N}^2}{\pi}|f_N|^2, \qquad \mu_{\chi N} = \frac{m_\chi m_N}{m_\chi + m_N}.$$

Using $f_N = -(y_\chi/(\Lambda m_h^2))m_N f^{(N)}$:

$$\boxed{\sigma_N^{\mathrm{SI}} = \frac{\mu_{\chi N}^2}{\pi}\left(\frac{y_\chi}{\Lambda m_h^2}\right)^2 m_N^2 \left(f^{(N)}\right)^2}$$

- Heavy DM: $\mu_{\chi N} \to m_N$, so σ_N^{SI} saturates.
- Event rate still decreases $\propto 1/m_\chi$ because $n_\chi = \rho_\chi/m_\chi$.

Scalar vs fermion Higgs portal: parametric scaling

Real scalar portal

$$\text{operator:} \quad -\lambda_p \chi^2 H^\dagger H \qquad \Rightarrow \qquad \sigma_N^{\mathrm{SI}} \propto \lambda_p^2 \, \frac{\mu_{\chi N}^2 m_N^2}{m_h^4}.$$

Fermion portal (dimension-5)

$$\text{operator:} \quad -\frac{y_\chi}{\Lambda} \bar{\chi}\chi H^\dagger H \qquad \Rightarrow \qquad \sigma_N^{\mathrm{SI}} \propto \left(\frac{y_\chi}{\Lambda}\right)^2 \, \frac{\mu_{\chi N}^2 m_N^2}{m_h^4}.$$

- Fermionic portal is suppressed by $1/\Lambda^2$.
- In UV completions, Λ is set by the mediator mass scale.

Pseudoscalar coupling (why direct detection is suppressed)

Consider a CP-odd coupling of fermionic DM to a mediator (written here as h):

$$\mathcal{L} \supset i g_\chi \, \bar{\chi}\gamma^5\chi \, h, \qquad \mathcal{L} \supset i \sum_q g_q \, \bar{q}\gamma^5 q \, h.$$

Key feature

In the non-relativistic limit, the pseudoscalar bilinear is momentum-suppressed:

$$\bar{u}(p')\gamma^5 u(p) \;\propto\; \mathcal{O}\left(\frac{|\vec{q}|}{m}\right).$$

Therefore the scattering amplitude is strongly suppressed for the small $|\vec{q}|$ relevant to direct detection.

Why $\bar{u}'\gamma^5 u$ is momentum-suppressed

Work in the Dirac basis. For a fermion of mass m and 3-momentum $\vec{p}$, the positive-energy spinor can be written (NR expanded) as

$$u(\vec{p}, s) \simeq \sqrt{E+m} \begin{pmatrix} \xi_s \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\xi_s \end{pmatrix} \simeq \sqrt{2m} \begin{pmatrix} \xi_s \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{2m}\,\xi_s \end{pmatrix}, \qquad E \simeq m + \frac{\vec{p}^{\,2}}{2m}.$$

The adjoint is $\bar{u} = u^\dagger \gamma^0$. In the Dirac basis,

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Define $u(\vec{p},s) = \begin{pmatrix}\phi\\ \chi\end{pmatrix}$ with $\phi \simeq \sqrt{2m}\,\xi$, $\chi \simeq \sqrt{2m}\,\frac{\vec{\sigma}\cdot\vec{p}}{2m}\xi$. Then

$$\bar{u}(\vec{p}',s')\,\gamma^5\,u(\vec{p},s) = (\phi'^{\dagger},\,-\chi'^{\dagger})\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \phi'^{\dagger}\chi\;-\;\chi'^{\dagger}\phi.$$

Using the NR forms of ϕ, χ,

$$\bar{u}(\vec{p}',s')\,\gamma^5\,u(\vec{p},s) \simeq \xi^{\dagger}_{s'}\,\vec{\sigma}\cdot(\vec{p}-\vec{p}')\,\xi_s = -\xi^{\dagger}_{s'}\,\vec{\sigma}\cdot\vec{q}\,\xi_s, \qquad \vec{q}\equiv\vec{p}'-\vec{p}.$$

Result

$$\boxed{\bar{u}(\vec{p}',s')\,\gamma^5\,u(\vec{p},s) = \mathcal{O}(|\vec{q}|)} \quad \Rightarrow \quad \text{pseudoscalar bilinear vanishes as } |\vec{q}| \to$$

NR scaling: pseudoscalar–pseudoscalar vs scalar–scalar

For a contact interaction $(\bar{\chi} i\gamma^5 \chi)(\bar{N} i\gamma^5 N)$ the amplitude is

$$\mathcal{A}_{\rm PS} \propto \left[\bar{u}'_\chi \gamma^5 u_\chi\right] \left[\bar{u}'_N \gamma^5 u_N\right] \sim \mathcal{O}(|\vec{q}|)\,\mathcal{O}(|\vec{q}|) \propto |\vec{q}|^2.$$

Therefore,

$$\boxed{|\mathcal{A}_{\rm PS}|^2 \propto |\vec{q}|^4 \quad \Rightarrow \quad \sigma_{\rm PS} \propto |\vec{q}|^4}$$

(up to kinematic prefactors).

Compare with the scalar operator $(\bar{\chi}\chi)(\bar{N}N)$:

$$\mathcal{A}_{\rm S} \propto \left[\bar{u}'_\chi u_\chi\right] \left[\bar{u}'_N u_N\right].$$

Using the same NR spinors,

$$\bar{u}(\vec{p}', s')\, u(\vec{p}, s) = (\phi'^\dagger, -\chi'^\dagger) \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \phi'^\dagger \phi - \chi'^\dagger \chi \simeq 2m\, \xi^\dagger_{s'} \xi_s + \mathcal{O}\left(\frac{\vec{p}^{\,2}}{m}\right),$$

so

$$\boxed{\bar{u}' u = \mathcal{O}(m) \quad \Rightarrow \quad \mathcal{A}_{\rm S} = \mathcal{O}(m_\chi m_N) \;\; \text{(unsuppressed as } |\vec{q}| \to 0\text{).}}$$

NR scaling of the amplitude (pseudoscalar–pseudoscalar)

For pseudoscalar exchange with vertices $(\bar{\chi} i\gamma^5 \chi)(\bar{q} i\gamma^5 q)$, the amplitude scales as

$$\mathcal{A} \propto \left[\bar{u}_\chi(p')\gamma^5 u_\chi(p)\right] \left[\bar{u}_q(k')\gamma^5 u_q(k)\right] \sim \mathcal{O}\left(\frac{|\vec{q}|}{m_\chi}\right) \mathcal{O}\left(\frac{|\vec{q}|}{m_N}\right) \propto |\vec{q}|^2.$$

Consequence

The cross section scales roughly as

$$\sigma \propto |\vec{q}|^4,$$

so it is highly suppressed at small recoil energies.

How strong is the suppression? (order-of-magnitude)

Typical direct-detection momentum transfer:

$$|\vec{q}| \sim 10\text{–}100 \text{ MeV}, \qquad m_N \sim 1 \text{ GeV}.$$

Hence,

$$\left(\frac{|\vec{q}|}{m_N}\right)^2 \sim 10^{-4}\text{–}10^{-2}, \qquad \left(\frac{|\vec{q}|}{m_N}\right)^4 \sim 10^{-8}\text{–}10^{-4}.$$

Takeaway

Even for order-one couplings, pseudoscalar-mediated scattering can be many orders of magnitude below the scalar SI case and may evade current direct-detection bounds.