

# Modelação Numérica 2017

## Aula 4, 22/Fev

- Álgebra complexa
- Circuito RLC
- Filtros

<http://modnum.ucs.ciencias.ulisboa.pt>

# Transformada de Fourier

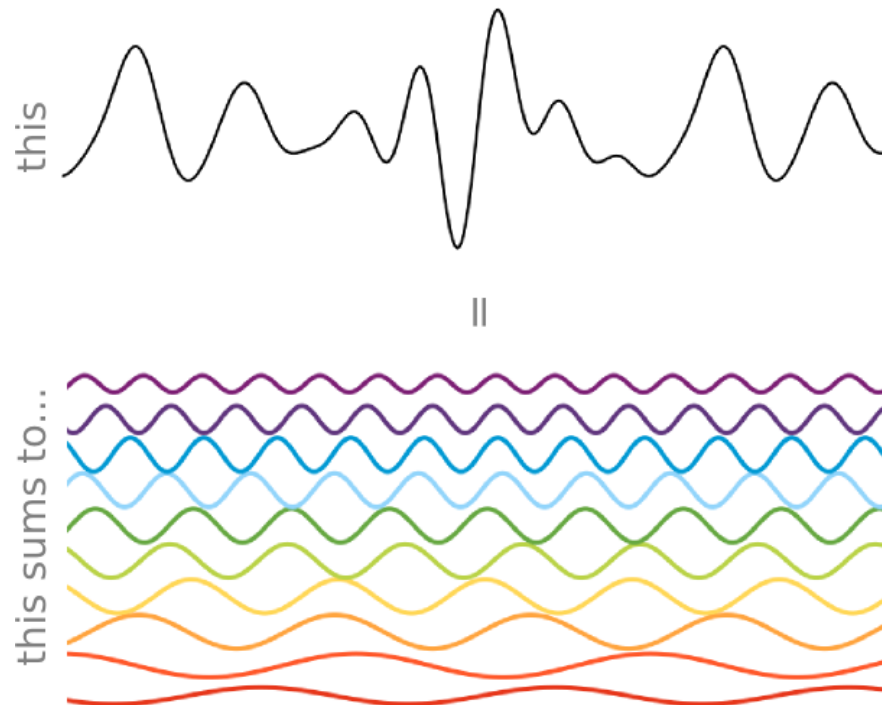


[https://en.wikipedia.org/wiki/File:Fourier\\_transform\\_time\\_and\\_frequency\\_domains\\_\(small\).gif](https://en.wikipedia.org/wiki/File:Fourier_transform_time_and_frequency_domains_(small).gif)

# Generalização a qualquer função (não senos)

- Qualquer função periódica pode ser representada como uma série de Fourier, i.e.:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right)$$



# Série de Fourier

$$\begin{aligned}
 f(t) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right) = \\
 &= \frac{a_0}{2} + \underbrace{\left( a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} \right)}_{\text{Primeira harmónica}} + \left( a_2 \cos \frac{2\pi t}{T/2} + b_2 \sin \frac{2\pi t}{T/2} \right) + \dots
 \end{aligned}$$

Primeira harmónica:

$$a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} = A \cos \left( \frac{2\pi t}{T} + \phi \right)$$

- $\Phi$ : Fase inicial
- $A$ : Amplitude
- $T$ : Período; Frequência  $f=1/T$ ; Frequência angular:  $\omega=2\pi f = 2\pi/T$

# Série de Fourier

$$\begin{aligned}
 f(t) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right) = \\
 &= \frac{a_0}{2} + \underbrace{\left( a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} \right)}_{\text{Primeira harmónica}} + \left( a_2 \cos \frac{2\pi t}{T/2} + b_2 \sin \frac{2\pi t}{T/2} \right) + \dots
 \end{aligned}$$

Primeira harmónica:

$$a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} = A \cos \left( \frac{2\pi t}{T} + \phi \right)$$

$$\begin{aligned}
 &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k - ib_k}{2} e^{i2\pi kt/T} + \sum_{k=1}^{\infty} \frac{a_k + ib_k}{2} e^{-i2\pi kt/T} = \\
 &= \sum_{k=-\infty}^{\infty} c_k e^{i2\pi kt/T}
 \end{aligned}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

# Transformada Discreta de Fourier

- Directa:

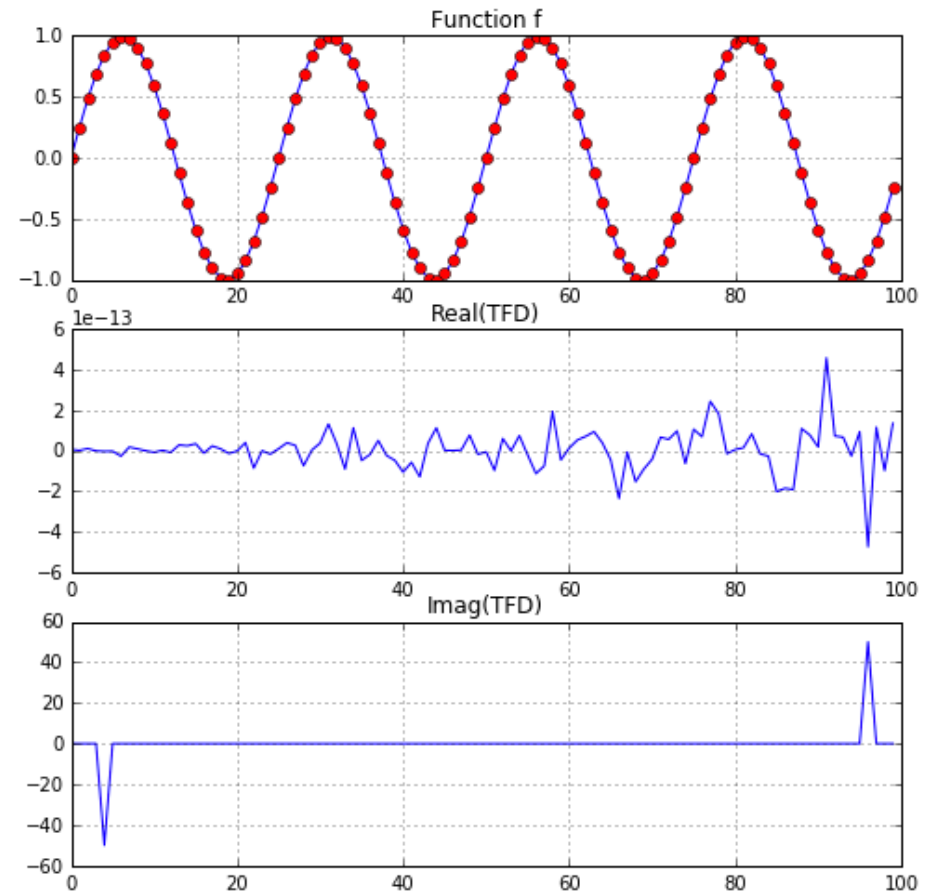
$$c_k = \sum_{t=0}^{N-1} f(t_n) e^{-i2\pi t_n k/N}$$

- Inversa:

$$f(t_n) = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{i2\pi t_n k/N}$$

$$F = \mathcal{F}(f)$$

$$f = \mathcal{F}^{-1}(F) = \mathcal{F}^{-1}(\mathcal{F}(f))$$



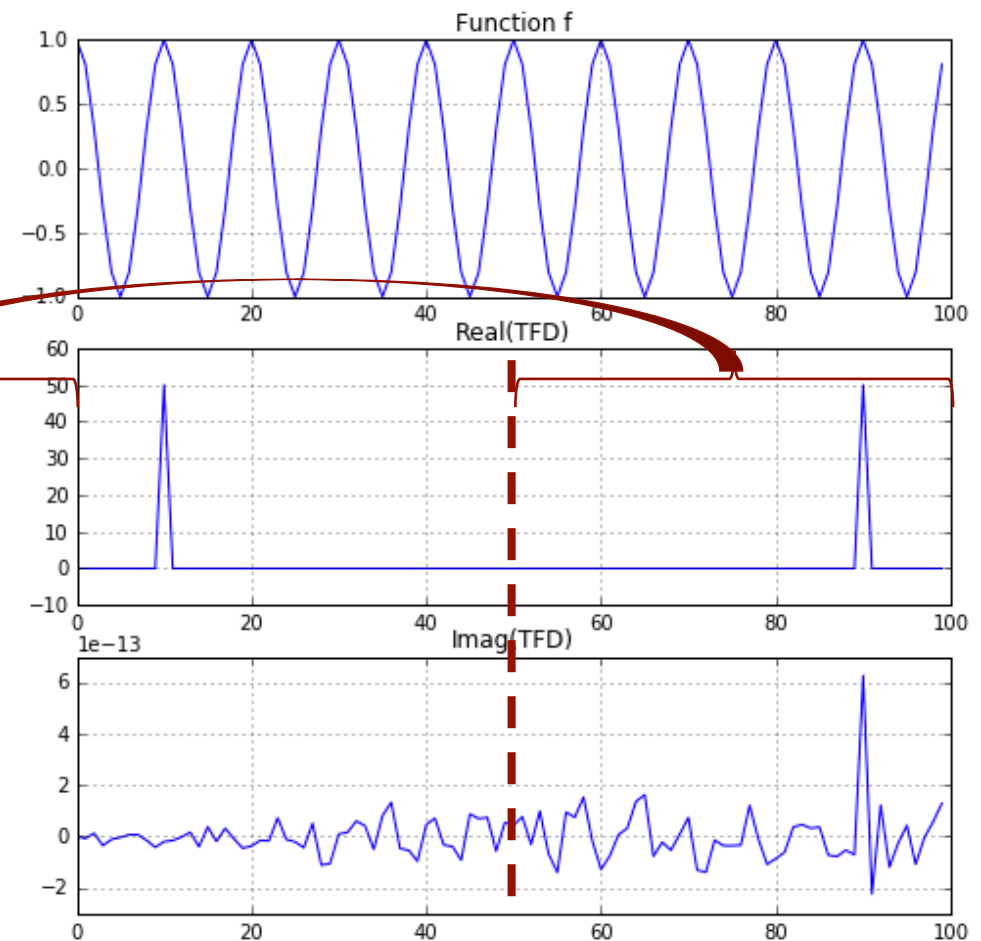
# Interpretação do espectro: Frequência

Coseno, 10 ciclos exactos,  $T = 10$  s,  $f = 0.1$  Hz

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi ft} dt$$

$$c_k = \frac{1}{2\pi} \int_{-T/2}^{T/2} f(t)e^{-i2\pi kt/T}, k = 0, \pm 1, \pm 2$$

$$c_k = \sum_{t=0}^{N-1} f(t_n)e^{-i2\pi t_n k/N}$$



# Interpretação do espectro: Frequência

Coseno, 10 ciclos exactos,  $T = 10$  s,  $f = 0.1$  Hz

```
N=100.; T=10.; dt=1.
t=np.arange(0.,N,dt)
f=np.cos(2*pi * t/T)
```

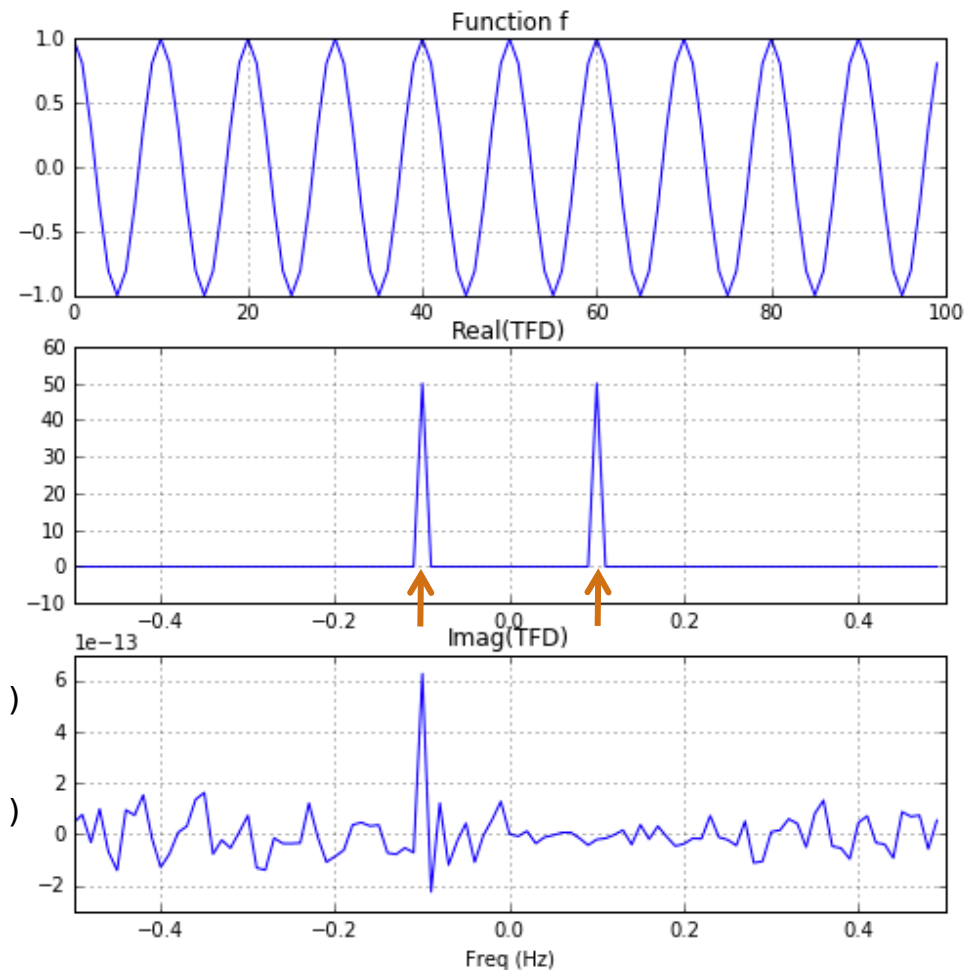
```
tf=TFD(f)
```

```
fout=np.concatenate([tf[N/2:], tf[:N/2]])
fNyq=1/(2*dt); df=1/(N*dt);
freq=np.arange(-fNyq, fNyq,df)
```

```
plt.close();
plt.subplot(3,1,1); plt.plot(t, f)
plt.title('Function f'); plt.grid();
```

```
plt.subplot(3,1,2); plt.plot(freq,np.real(fout))
plt.title('Real(TFD)'); plt.grid();
```

```
plt.subplot(3,1,3); plt.plot(freq,np.imag(fout))
plt.title('Imag(TFD)'); plt.grid();
```



Re-ordenando, obtém-se o espectro usual

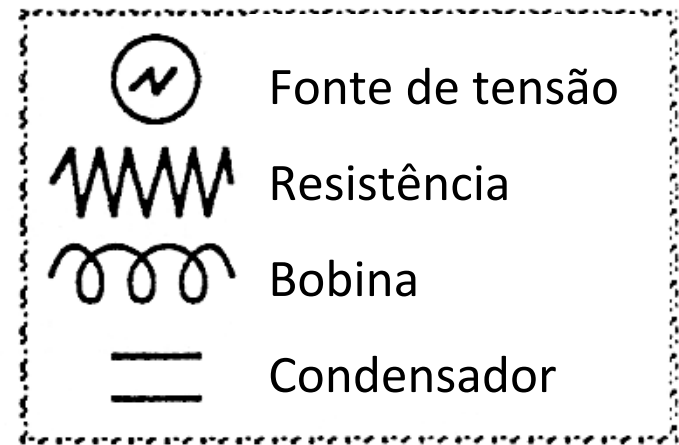
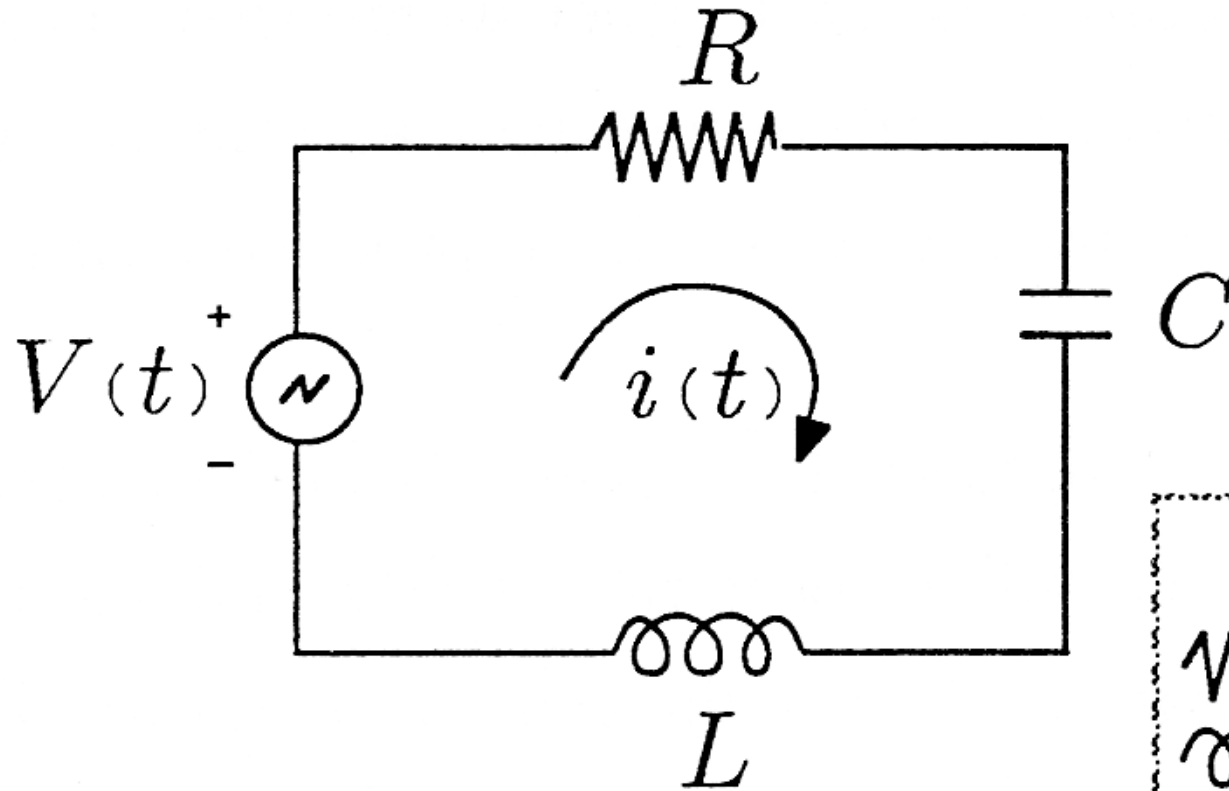


# Filtro Digital

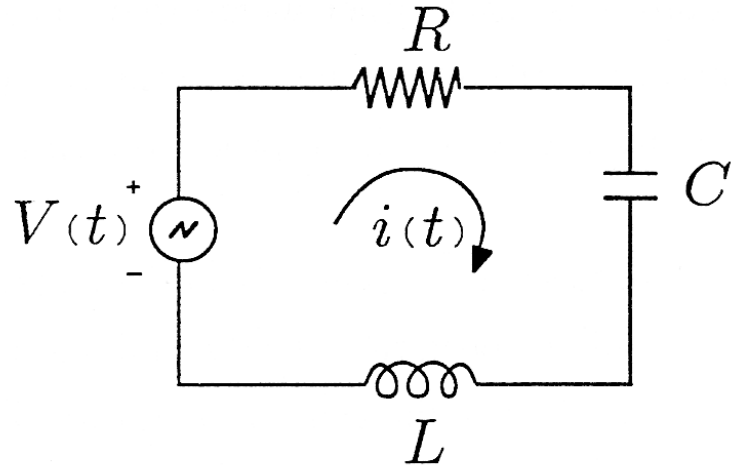


- **Filtros analógicos:** ex: circuito RLC.
- **Filtros lineares digitais:**
  - **Não recursivos:** o output em cada instante é uma combinação linear do input em instantes anteriores (fácil de analisar com análise de Fourier).
  - **Recursivos:** o output é uma combinação linear do input e do output em instantes anteriores.

# Circuito RLC (Projecto 0)



# Circuito RLC



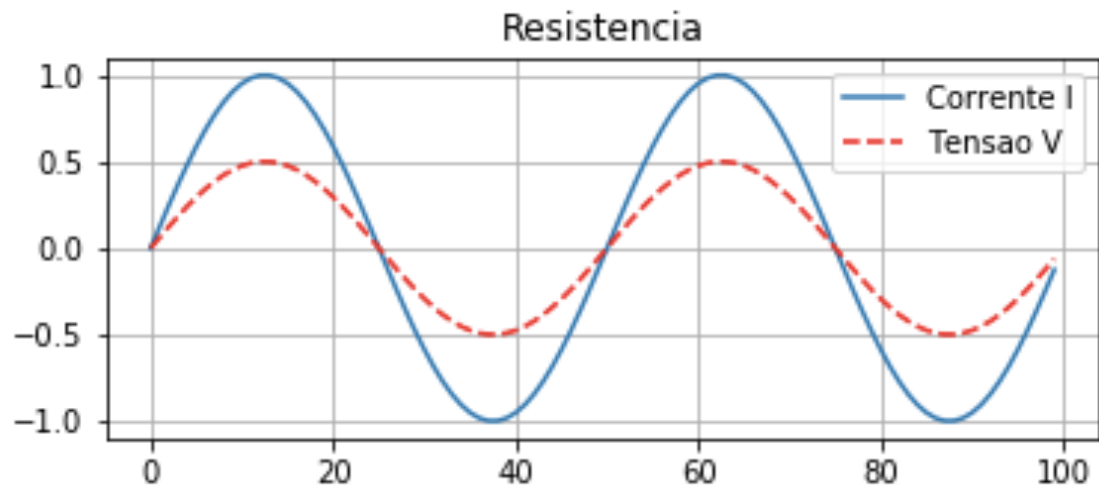
- Componentes:

$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

- Corrente alterna:  $I = I_0 \sin(\omega t)$

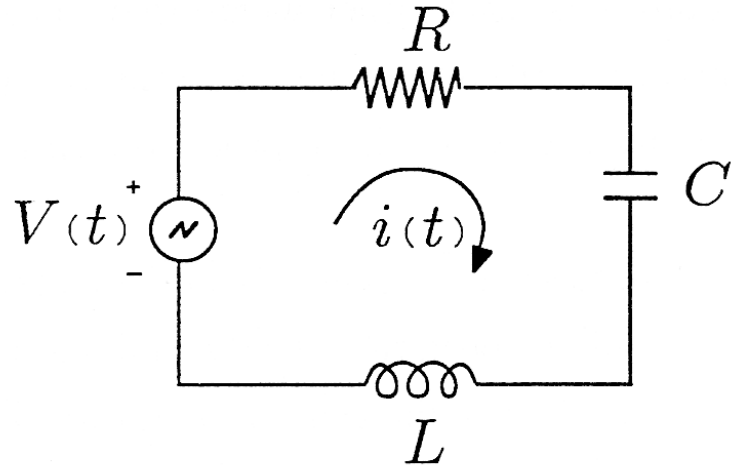
Resistência:

$$\begin{aligned} V_R &= RI \\ &= RI_0 \sin(\omega t) \end{aligned}$$



V e I em fase, resposta depende de R.

# Circuito RLC



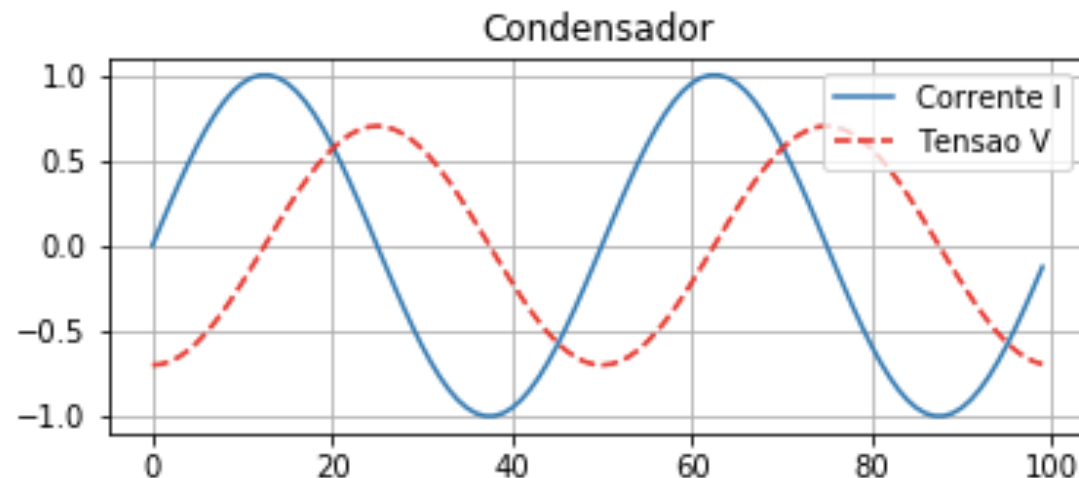
- Componentes:

$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

- Corrente alterna:  $I = I_0 \sin(\omega t)$

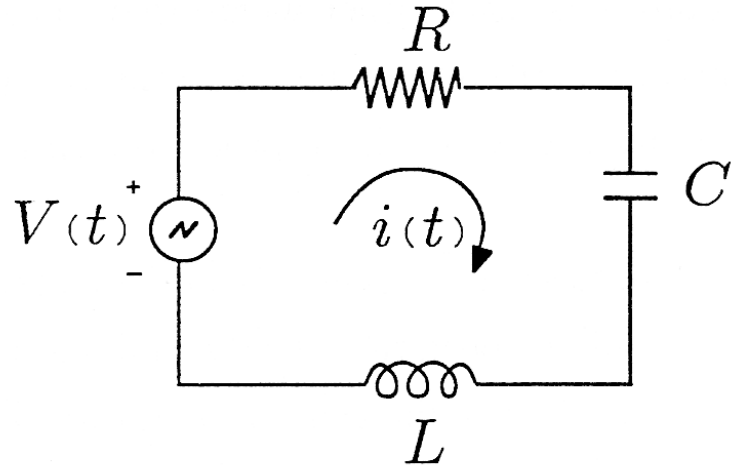
## Condensador:

$$\begin{aligned} V_C &= \frac{1}{C} \int I dt \\ &= \frac{1}{C} \int I_0 \sin(\omega t) dt \\ &= \frac{1}{C} \frac{I_0}{\omega} (-\cos(\omega t)) \end{aligned}$$



V e I desfasados, V atrasado em relação a I, resposta depende da frequência ( $1/\omega$ , passa-alto).

# Circuito RLC



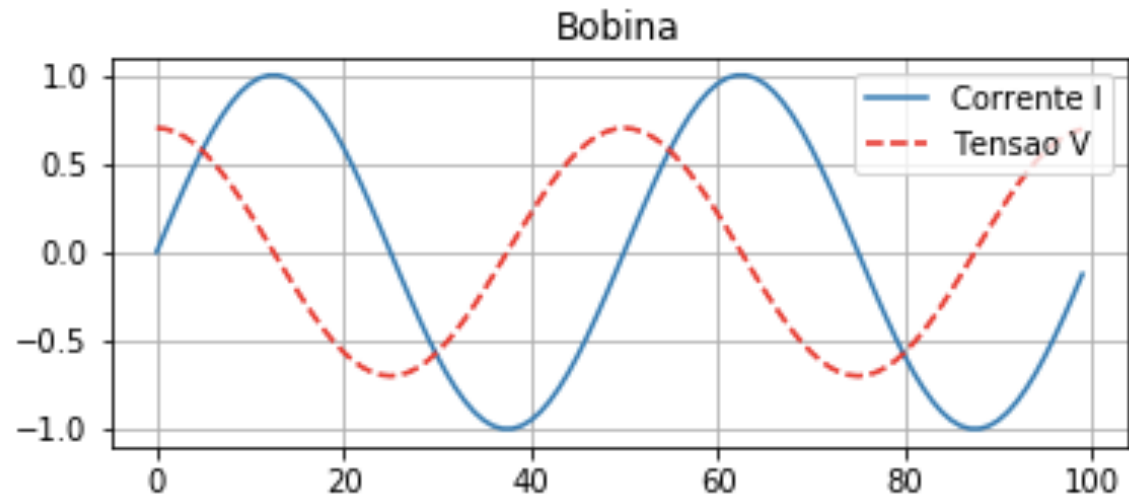
- Componentes:

$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

- Corrente alterna:  $I = I_0 \sin(\omega t)$

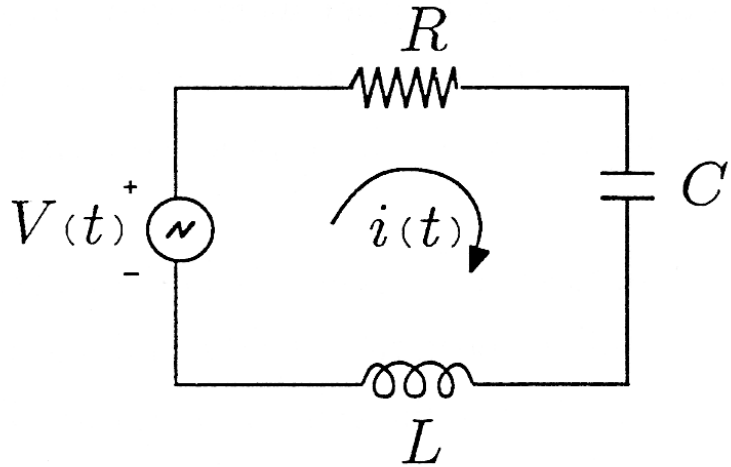
Bobina:

$$\begin{aligned} V_L &= L \frac{dI}{dt} \\ &= L \frac{d}{dt} I_0 \sin(\omega t) \\ &= LI_0 \omega \cos(\omega t) \end{aligned}$$



V e I desfasados, V adiantado em relação a I, resposta depende da frequência ( $\omega$ , passa-baixo).

## Circuito RLC — Em conjunto...



- Componentes lineares:

$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

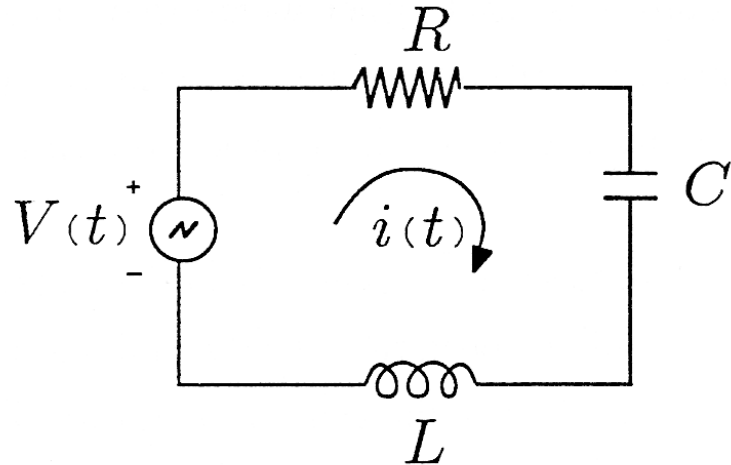
Lei de Kirchoff das malhas + tensão sinusoidal:

$$V = V_L + V_C + V_R = V_0 \cos(\omega t)$$

Como todos os componentes têm um comportamento linear:

$$I = I_0 \cos(\omega t + \phi)$$

# Circuito RLC — Utilizando a álgebra complexa



- Componentes lineares:

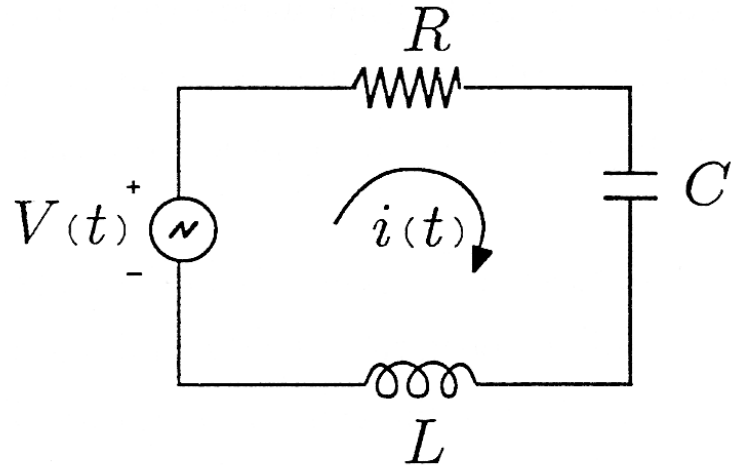
$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

$$\begin{cases} V = V_0 \cos(\omega t) \\ I = I_0 \cos(\omega t + \phi) \end{cases}$$

Diferenciação no domínio do tempo corresponde a multiplicar por  $\omega$  no domínio espectral

$$\begin{cases} V_R = RI = RI_0 e^{i(\omega t + \phi)} \\ V_C = \frac{1}{C} \int I dt = \frac{1}{C} \int I_0 e^{i(\omega t + \phi)} dt = \frac{I_0}{C} \frac{1}{i\omega} e^{i(\omega t + \phi)} = -\frac{i}{\omega C} I_0 e^{i(\omega t + \phi)} \\ V_L = L \frac{dI}{dt} = L \frac{d}{dt} (I_0 e^{i(\omega t + \phi)}) = Li\omega I_0 e^{i(\omega t + \phi)} \end{cases}$$

# Circuito RLC — Utilizando a álgebra complexa



- Componentes lineares:

$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

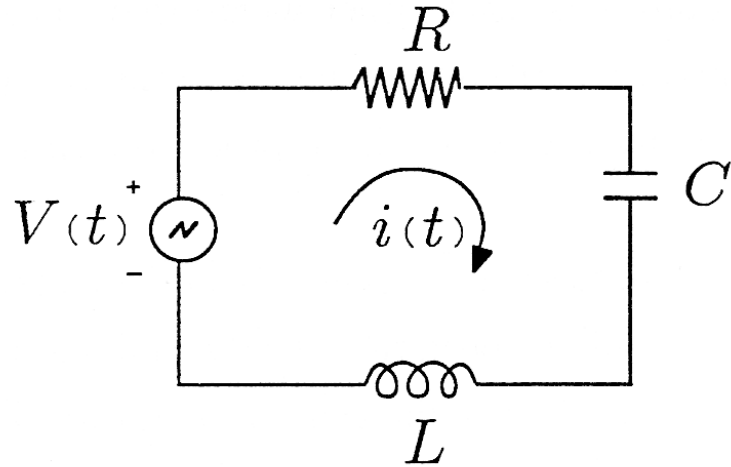
$$\begin{cases} V = V_0 \cos(\omega t) \\ I = I_0 \cos(\omega t + \phi) \end{cases}$$

Integração no domínio do tempo  
corresponde a dividir por  $\omega$  no  
domínio espectral

$$\begin{cases} V_R = RI = RI_0 e^{i(\omega t + \phi)} \\ V_C = \frac{1}{C} \int I dt = \frac{1}{C} \int I_0 e^{i(\omega t + \phi)} dt = \frac{I_0}{C} \frac{1}{i\omega} e^{i(\omega t + \phi)} = -\frac{i}{\omega C} I_0 e^{i(\omega t + \phi)} \\ V_L = L \frac{dI}{dt} = L \frac{d}{dt} (I_0 e^{i(\omega t + \phi)}) = Li\omega I_0 e^{i(\omega t + \phi)} \end{cases}$$



# Circuito RLC — Utilizando a álgebra complexa



- Componentes lineares:

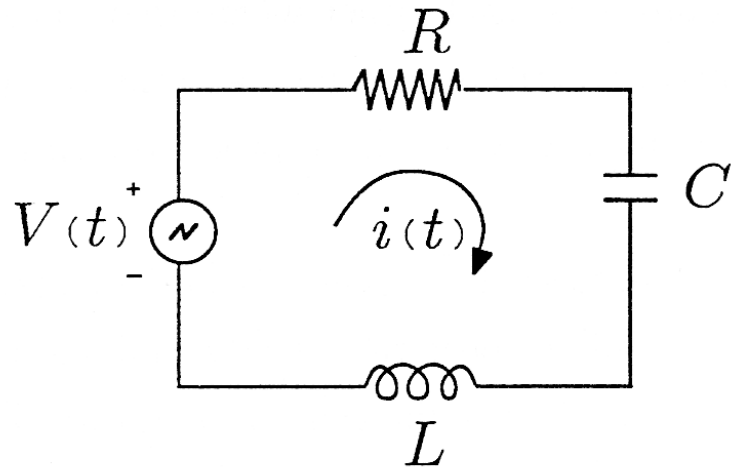
$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

$$\begin{cases} V = V_0 \cos(\omega t) \\ I = I_0 \cos(\omega t + \phi) \end{cases}$$

- Linearidade input-output
- Desfasamento
- Resposta depende da frequência

$$\begin{cases} \underline{V_R} = RI = \underline{RI_0 e^{i(\omega t + \phi)}} \\ \underline{V_C} = \frac{1}{C} \int I dt = \frac{1}{C} \int I_0 e^{i(\omega t + \phi)} dt = \frac{I_0}{C} \frac{1}{i\omega} e^{i(\omega t + \phi)} = -\frac{i}{\omega C} \underline{I_0 e^{i(\omega t + \phi)}} \\ \underline{V_L} = L \frac{dI}{dt} = L \frac{d}{dt} (I_0 e^{i(\omega t + \phi)}) = \underline{i\omega L I_0 e^{i(\omega t + \phi)}} \end{cases}$$

## Circuito RLC — Em conjunto...



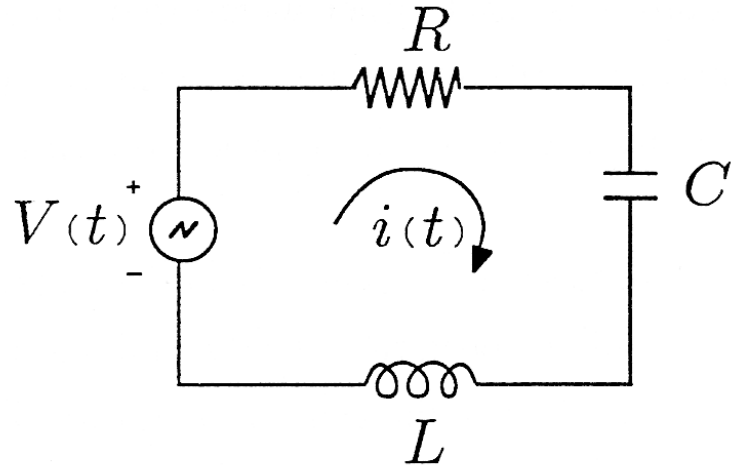
$$V = V_L + V_C + V_R = V_0 \cos(\omega t)$$

$$V = V_0 e^{i\omega t} = \left[ R - \frac{i}{\omega C} + i\omega L \right] I_0 e^{i(\omega t + \phi)}$$

Impedância complexa Z

(resposta em amplitude e fase)

# Circuito RLC — Em conjunto...



$$\begin{cases} V_R = RI \\ V_C = \frac{1}{C} \int I dt \\ V_L = L \frac{dI}{dt} \end{cases}$$

A equação diferencial pode resolver-se como (mais simples) uma equação aritmética

$$V = V_L + V_C + V_R = V_0 \cos(\omega t)$$

$$V = V_0 e^{i\omega t} = \left[ R - \frac{i}{\omega C} + i\omega L \right] I_0 e^{i(\omega t + \phi)}$$

$$V_0 = \left[ R - \frac{i}{\omega C} + i\omega L \right] I_0 e^{i\phi} \Leftrightarrow \boxed{\frac{V_0}{R - \frac{i}{\omega C} + i\omega L} = I_0 e^{i\phi}}$$

# Circuito RLC — Exemplo

$$\frac{V_0}{R - \frac{i}{\omega C} + i\omega L} = I_0 e^{i\phi}$$

```
# Parâmetros:  
R=1000.      # Resistência, ohm  
L=1.0e-2    # Impedância da Bobine, H  
C=1.0e-6    # Capacidade do condensador, F  
V0=1.0      # Amplitude do potencial aplicado,
```

```
## Cálculo:
```

```
omega = 2*pi* np.arange(10., 1.e6, 10.) #
```

```
Z = R - 1j/(omega*C) + 1j*omega*L ;
```

```
I=V0/Z
```

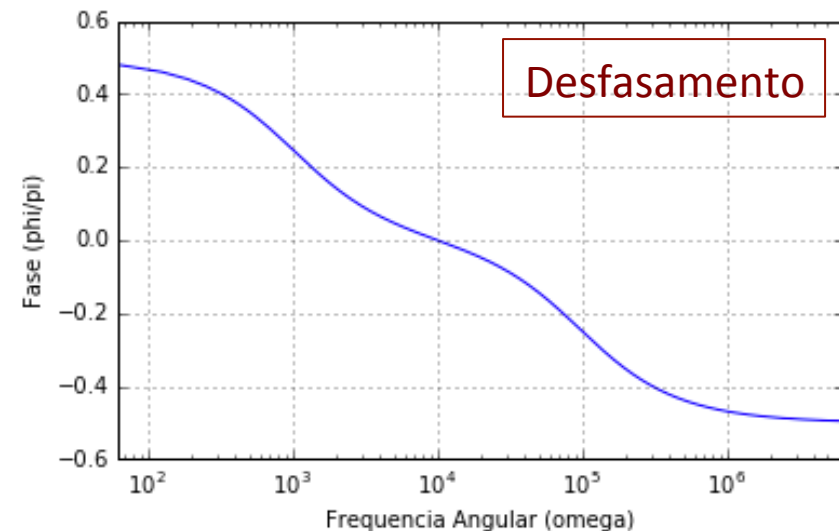
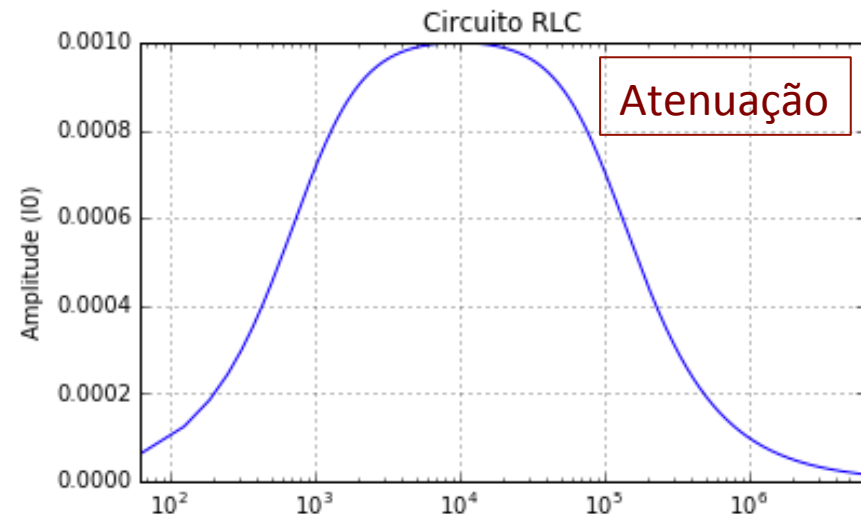
```
phi=np.angle(I)
```

```
I0=np.abs(I)
```

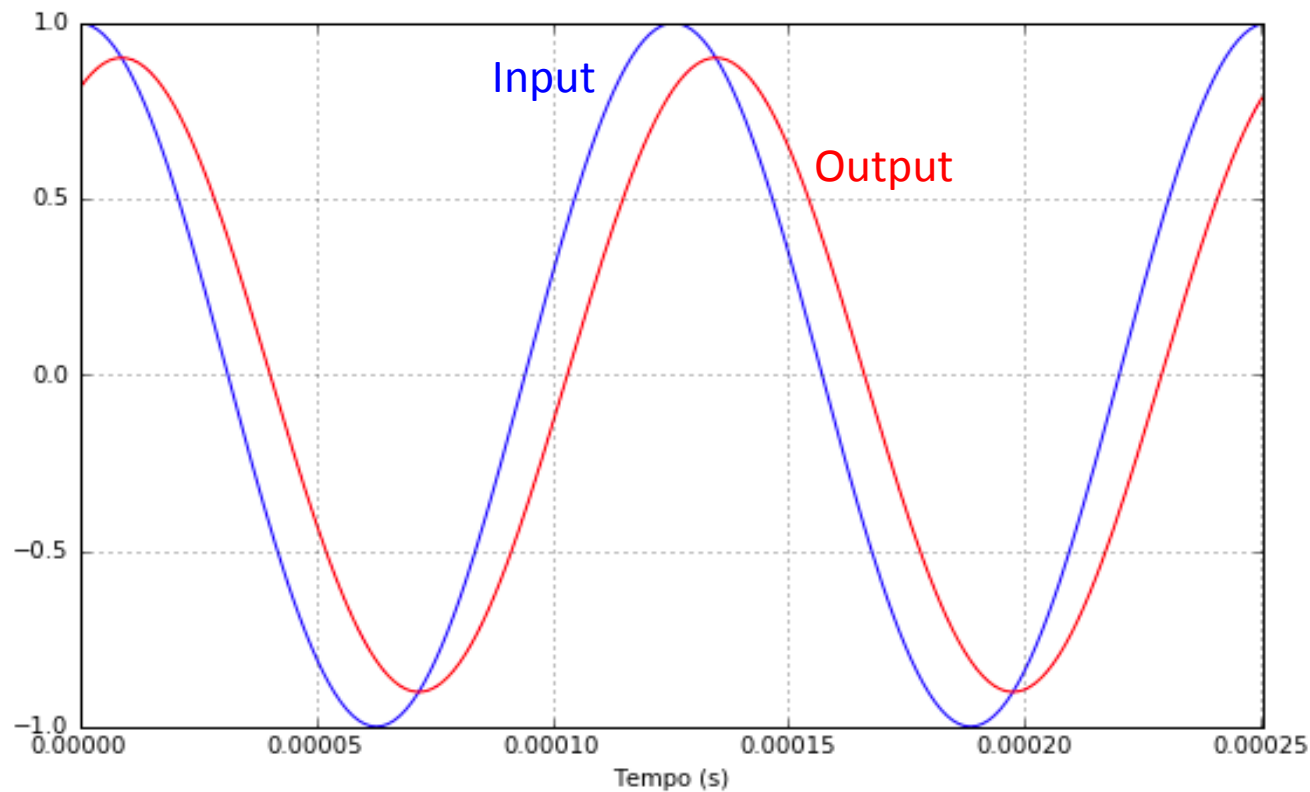
```
# Gráficos:
```

```
plt.subplot(2,1,1)  
plt.semilogx(omega,I0)  
plt.title('Circuito RLC')  
plt.ylabel('Amplitude (I0)')  
plt.autoscale(enable=True, axis='x', tight=True)  
plt.grid()
```

```
plt.subplot(2,1,2)  
plt.semilogx(omega,phi/pi)  
plt.xlabel('Frequencia Angular (omega)')  
plt.ylabel('Fase (phi/pi)')
```



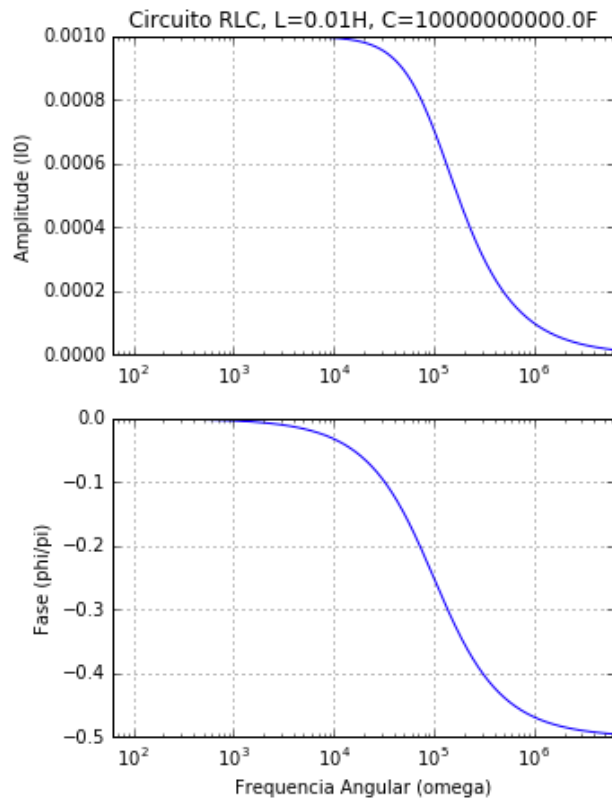
# Circuito RLC — Exemplo



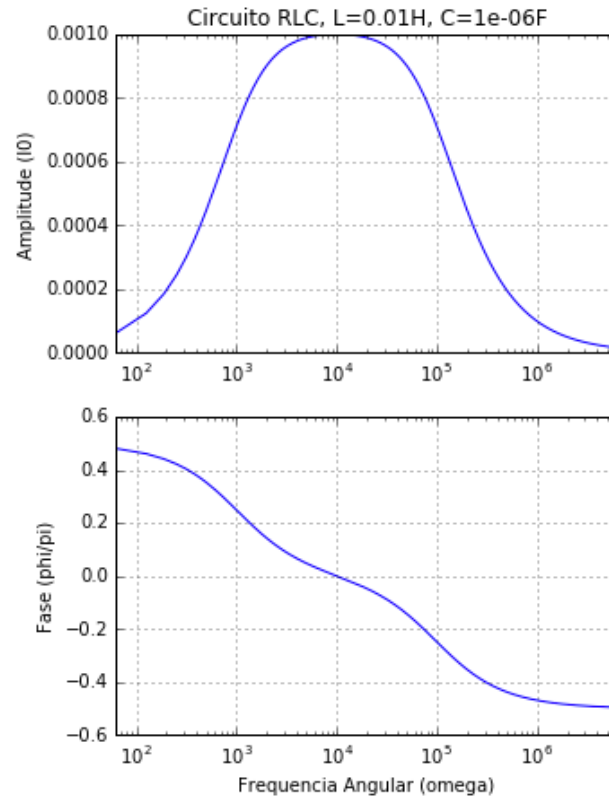
- Linearidade input-output: Input e output são senos de iguais frequências.
- Desfasamento: O output está ligeiramente atrasado em relação ao input.
- Atenuação: O output tem menor amplitude que o input.

# Circuito RLC — Variando L e C

## Filtro passa-baixo



## Filtro passa-banda



## Filtro passa-alto

