CLP

Lecture 9

9 Similarity Theory

Homework 1 (Exam style)

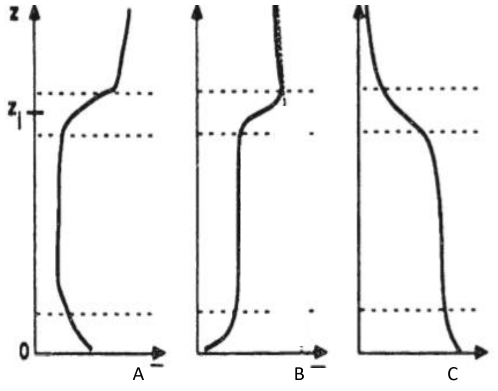
1. Explain in less than five lines:

a. Virtual potential temperature and its relevance.

b. The existence and properties of the “residual layer”

c. When can we accept the hydrostatic approximation?

d. The following figure represents three mean variables in a boundary layer taken from the list: temperature, potential temperature, specific humidity, relative humidity, wind speed. (a) identify A,B,C justifying your choice; (b) classify the Boundary Layer.



Homework 2 (Exam style)

2. Consider the term $-u_i \frac{\partial q}{\partial x_i}$, where q is specific humidity. Use the Reynolds decomposition and averaging to expand into mean and turbulent sub-terms. Simplify in conditions applicable to boundary layer flow.

3. The Ekman (spiral) solution describes the wind in the boundary layer in the form:

$$u = u_g(1 - e^{-\gamma z}\cos(\gamma z))$$

$$v = u_g e^{-\gamma z}\sin(\gamma z)$$

a. List the assumptions taken to arrive to that solution.

b. Defining the boundary layer height as $z = \pi/\gamma$, prove the existence of "Ekman pumping" (mass transport across isobars towards low pressure).

Homework 3 (Exam style)

4. Consider the equation:

$$\frac{\partial \bar{e}}{\partial t} = \frac{g}{\overline{\theta_v}} \overline{w'\theta_v'} - \overline{u'w'} \frac{\partial \bar{u}}{\partial z} - \frac{1}{\bar{\rho}} \frac{\partial \overline{w'p'}}{\partial x} - \frac{\partial \overline{w'e'}}{\partial z} - \varepsilon$$

a. Explain the different terms.

b. Which terms are always negative?

c. Which are always positive?

5. Assume that variation of TKE (turbulent kinetic energy) with z (height above ground) is controlled by the following variables: $g, \theta_s, z_i, \overline{w'\theta'}_s$.

a. How many PI groups can be defined for this problem?

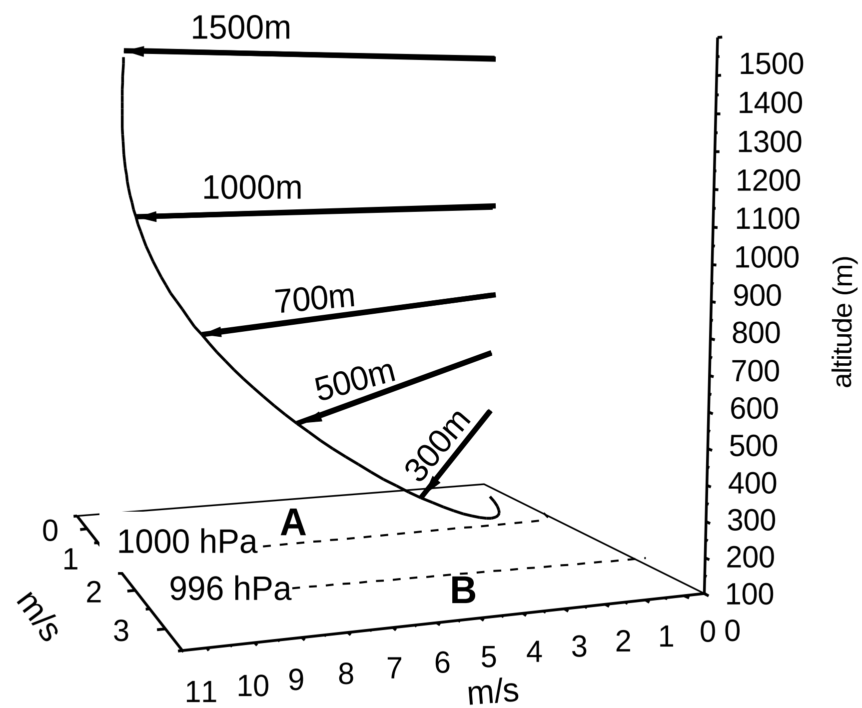
b. Compute the PI groups using the control variables as key variables.

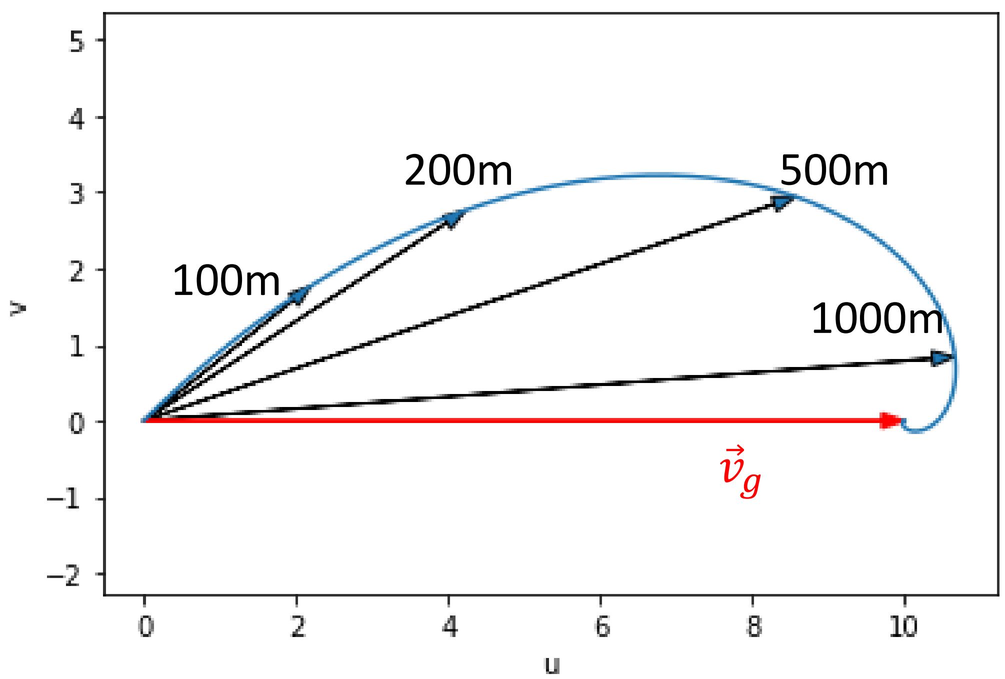
Ekman ($K_m = const$)

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + f\bar{v} - \frac{\partial}{\partial z}\left(-K_m \frac{\partial \bar{u}}{\partial z}\right) = 0$$

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - f\bar{u} - \frac{\partial}{\partial z}\left(-K_m \frac{\partial \bar{v}}{\partial z}\right) = 0$$

$$\begin{cases} u = u_g(1 - e^{-\gamma z}\cos\gamma z) \\ v = u_g e^{-\gamma z}\sin\gamma z \end{cases}$$





No change in direction for z<100m

The surface layer

Is a “constant” flux layer (turbulent+molecular)

$Z<100$m

Constant flux layer, but not constant mixing (as eddies tend to zero at the surface)

$$\overline{w'u'} = -K_m \frac{\partial \bar{u}}{\partial z}$$

$$-\frac{\partial}{\partial z}\left(-K_m \frac{\partial \bar{u}}{\partial z}\right) = 0 \Rightarrow \left(K_m \frac{\partial \bar{u}}{\partial z}\right) = const$$

If K_m is proportional to z near the surface, implying $\lim_{z \to 0} K = 0$:

$$kz \frac{\partial \bar{u}}{\partial z} = u_*$$

k –von Karman constant

Integrating:

$$\bar{u} = \frac{u_*}{k} ln\left(\frac{z}{z_0}\right)$$

Where z_0 is the roughness length.

Flat sea: 0.0002 m

Snow: 0.005 m

Grass: 0.03 m

Shrubs: 0.1 m

Forest, scattered houses: 1m

City: >2 m

z_0

Lettau (1969) suggested a method for estimating the aerodynamic roughness length based on the average vertical extent of the roughness elements (h*), the average silhouette or vertical cross-section area presented to the wind by one element (s_s), and the lot size per element [S_L = (total ground surface area / number of elements)]

$$z_o = 0.5\, h^* \left(\frac{s_s}{S_L} \right) \qquad (9.7.2a)$$

z_0

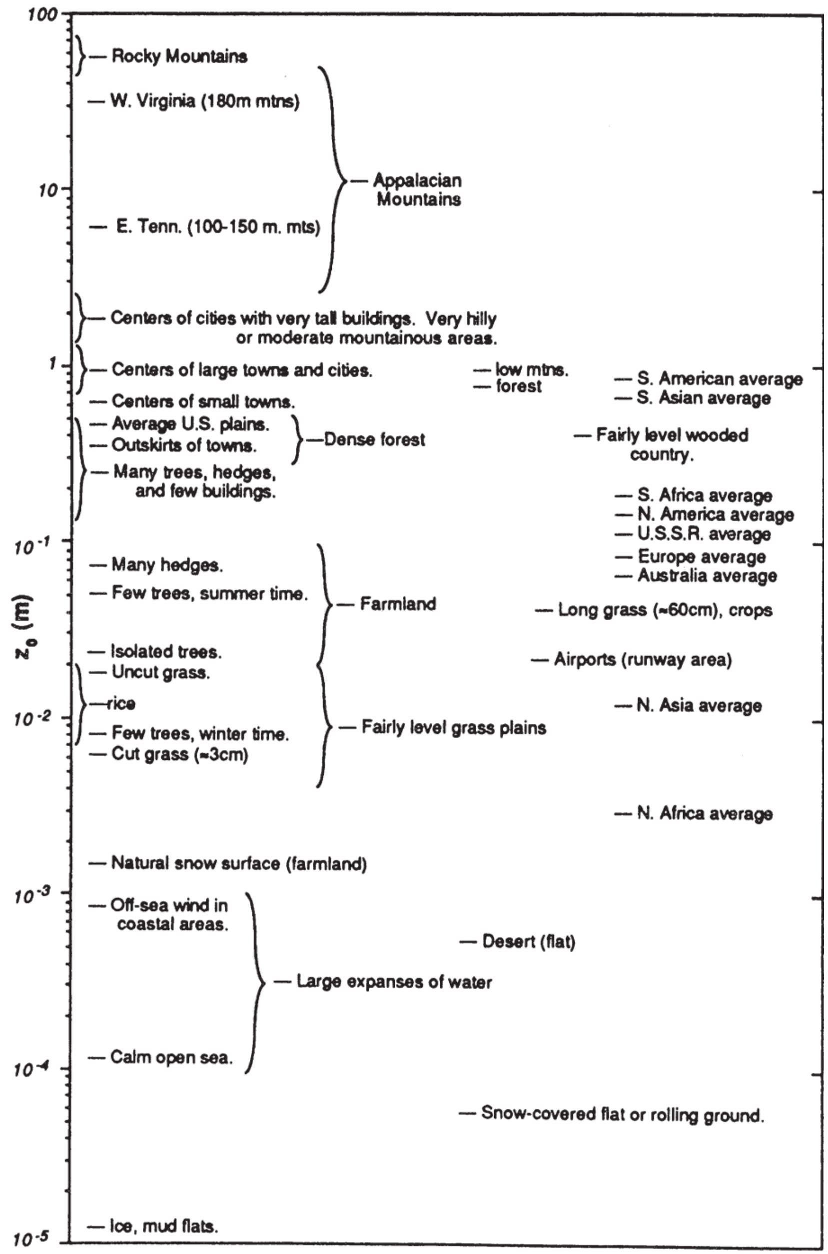


Fig. 9.6 Aerodynamic roughness lengths for typical terrain types. (After Garratt 1977, Smedman-Högström & Högström 1978, Kondo &

Displacement distance

9.7.3 Displacement Distance

Over land, if the individual roughness elements are packed very closely together, then the top of those elements begins to act like a displaced surface. For example, in some forest canopies the trees are close enough together to make a solid-looking mass of leaves, when viewed from the air. In some cities the houses are packed close enough together to give a similar effect; namely, the average roof-top level begins to act on the flow like a displaced surface.

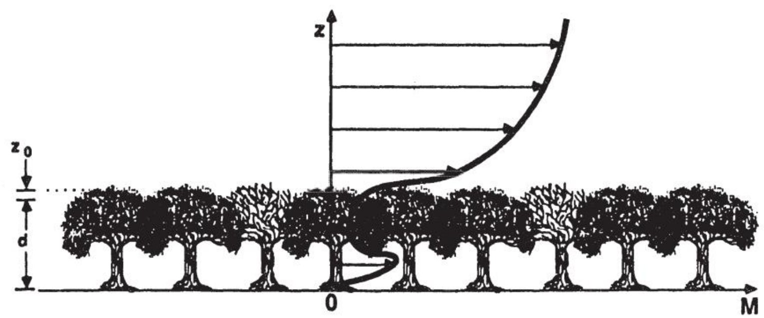


Fig. 9.7 Flow over forest canopy showing wind speed, M, as a function of height, z. The thick canopy layer acts like a surface displaced a distance, d, above the true surface. z_0 = roughness length.

Above the canopy top, the wind profile increases logarithmically with height, as shown in Fig 9.7. Thus, we can define both a displacement distance, d, and a roughness length, z_o, such that:

General logarithmic profile

$$\overline{M} = \left(\frac{u_*}{k}\right) \ln\left[\frac{(z - d)}{z_o}\right] \qquad (9.7.3a)$$

for statically neutral conditions, where we now define $\overline{M} = 0$ at $z = d + z_o$. Given wind speed observations in statically neutral conditions at three or more heights, it is easy to use computerized non-linear regression algorithms such as the Marquardt Method or the Gauss-Newton Method to solve for the three parameters, u_*, z_o, and d.

9.3.4 Classes of Similarity Scales

The most common classes of similarity scaling are ***Monin-Obukhov similarity***, ***mixed-layer similarity***, ***local similarity***, ***local free convection***, and ***Rossby-number similarity***. When dealing with one of these well-defined classes of problems, it is appropriate to use the associated scaling variables as the key variables in a dimensional analysis.

Monin-Obukhov Similarity. This class is usually applied to the surface layer (Monin and Obukhov, 1954; Wyngaard, 1973; Sorbjan, 1986), and hence is sometimes called ***surface-layer similarity***. Earlier we defined the surface layer as that part of the boundary layer where the fluxes vary by less than 10% of their magnitude with height. To a first order approximation, this layer is a ***constant flux layer***. We can thus simplify our description of the surface layer by utilizing the flux at just one height — usually the surface.

Monin-Obhukov length

Relevant variables:

$$\left(\overline{w'\theta_v'}\right)_s \equiv Kms^{-1}$$

$$\overline{w'u'}_s = u_*^2 \equiv m^2s^{-2}$$

$$\bar{\theta}_v \equiv K, g \equiv ms^{-2}, z \equiv m$$

5 variables
3 dimensions

2 PI numbers

Length scale:

$$L = -\frac{u_*^3\bar{\theta}_v}{kg\left(\overline{w'\theta_v'}\right)_s}$$

Monin-Obhukov (constant flux)

Monin-Obukhov similarity works only when the winds are not calm, and u_* not zero. Relevant scales based on these surface fluxes and their typical orders of magnitude are listed here:

L	Order (1 m to 200 m)
z_o	Order (1 mm to 1 m)
u_*	Order (0.05 to 0.3 m/s)
θ_*^{SL}	Order (0.1 to 2.0 °C)
q_*^{SL}	Order (0.1 to 5 g_{water}/kg_{air})

Scales for pollutant concentration can be patterned after the humidity scale. Lists of Monin-Obukhov similarity relationships are tabulated in Sections 9.4 to 9.6, and a more detailed analysis of the log-wind profile in the surface layer is given in Section 9.7.

Mixed-layer similarity

Mixed-Layer Similarity. This class is applied to mixed layers that are in a state of free convection (Deardorff, 1972; Deardorff, et al., 1980; Sorbjan, 1986), assuming calm or light winds. Free convection conditions can occur during cold air advection over a warmer surface, or with solar heating of the land during the daytime in light wind conditions. The relevant scales and typical orders of magnitude for the mixed layer are:

z_i	Order (0.2 to 2 km)
w_*	Order (2 m/s)
θ_*^{ML}	Order (0.1 K)
q_*^{ML}	Order (0.1 g/m^3)
u_*^{ML}	Order (0.02 m/s)

Other scales, such as for pollutant concentration, can be defined in analogy to the moisture scale. More details are discussed in Section 9.6.

Neutral Boundary Layer Similarity Relationship

Surface-Layer Parameterizations:

$$\frac{k\,z}{u_*}\frac{\partial \overline{U}}{\partial z} = 1 \qquad (9.5.1c)$$

$$\frac{k\,z}{\theta_*^{SL}}\frac{\partial \overline{\theta}}{\partial z} = 0.74 \qquad (9.5.1d)$$

Neutral $\overline{w'\theta'_v}_s = 0 \Longrightarrow L = \infty$

Stability effects (through L): convective

Surface-Layer Relationships:

$$\phi_M = \frac{k\,z}{u_*}\frac{\partial \overline{U}}{\partial z} = \left(1 - 15\,\frac{z}{L}\right)^{-1/4} \qquad (9.6.1e)$$

Type equation here.

$$\phi_H = \frac{k\,z}{\theta_*^{SL}}\frac{\partial \overline{\theta}}{\partial z} = 0.74\left(1 - 9\,\frac{z}{L}\right)^{-1/2} \qquad (9.6.1f)$$

Neutral $L = \infty$x

Stability effects (through L): stable

$$\phi_M = \frac{kz}{u_*}\left[\left(\frac{\partial \overline{U}}{\partial z}\right)^2 + \left(\frac{\partial \overline{V}}{\partial z}\right)^2\right]^{1/2} = 1 + 4.7\frac{z}{L} \qquad (9.4.1c)$$

$$\phi_H = \frac{kz}{\theta_*^{SL}}\frac{\partial \overline{\theta}}{\partial z} = 0.74 + 4.7\frac{z}{L} \qquad (9.4.1d)$$

Neutral $L = \infty$

Non-neutral

Businger-Dyer Relationships. In non-neutral conditions, we might expect that the buoyancy parameter and the surface heat flux are additional relevant variables. When these are used with the variables from Section 9.7.1, Buckingham Pi analysis gives us three dimensionless groups (neglecting the displacement distance for now): $\overline{M}/u_*$, z/z_o, and z/L, where L is the Obukhov length. Alternatively, if we consider the shear instead of the speed, we get two dimensionless groups: ϕ_M and z/L. Based on field experiment data, Businger, et al., (1971) and Dyer (1974) independently estimated the functional form to be:

$$\phi_M = 1 + \left(\frac{4.7\,z}{L}\right) \quad \text{for } \frac{z}{L} > 0 \quad \text{(stable)} \qquad (9.7.5a)$$

$$\phi_M = 1 \quad \text{for } \frac{z}{L} = 0 \quad \text{(neutral)} \qquad (9.7.5b)$$

$$\phi_M = \left[1 - \left(\frac{15z}{L}\right)\right]^{-1/4} \quad \text{for } \frac{z}{L} < 0 \quad \text{(unstable)} \qquad (9.7.5c)$$

Businger-Dyer

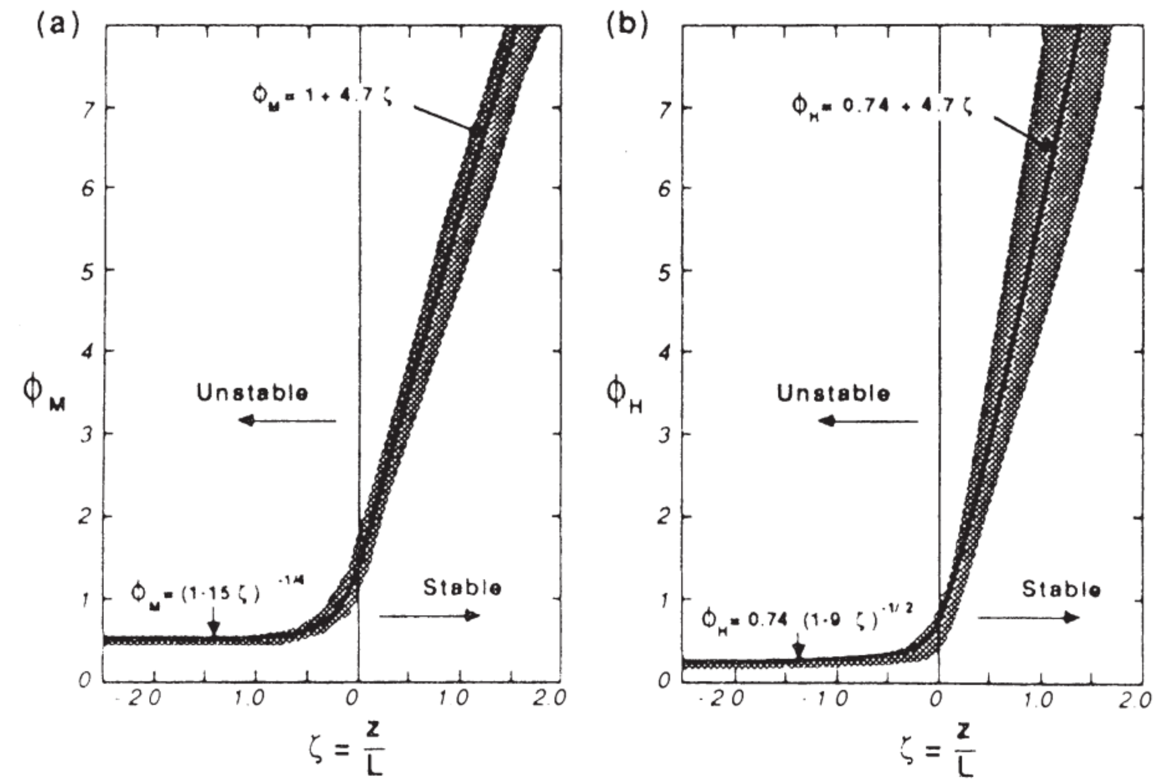
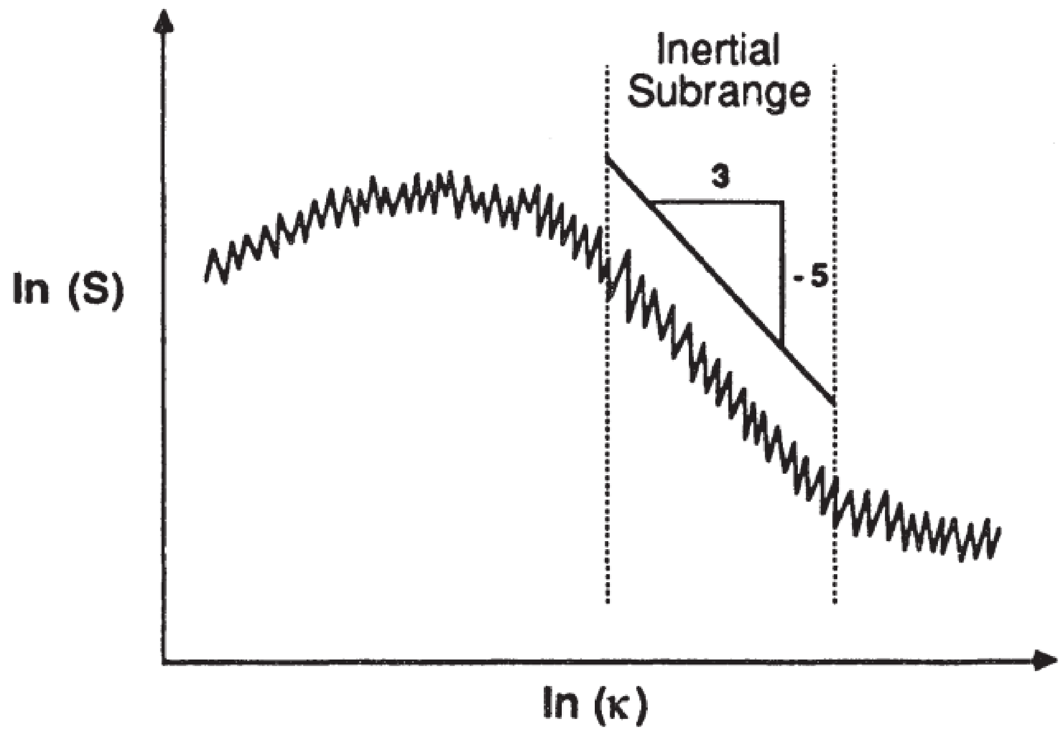


Fig. 9.9 (a) Range of dimensionless wind shear observations in the surface layer, plotted with interpolation formulas. (b) Range of dimensionless temperature gradient observations in the surface layer, plotted with interpolation formulas. After Businger, et al. (1971).

Spectral similarity

Fig. 9.12
On a log-log spectral plot, the inertial subrange appears as a straight line with -5/3 slope.



As discussed in Chapter 5, there are many situations where middle size turbulent eddies "feel" neither the effects of viscosity, nor the generation of TKE. These eddies get their energy inertially from the larger-size eddies, and lose their energy the same way to smaller-size eddies. For a steady-state turbulent flow, the cascade rate of energy down the spectrum must balance the dissipation rate at the smallest eddy sizes. Hence, there are only three variables relevant to the flow: S, κ, and ε. This similarity approach was pioneered by Kolmogorov (1941) and Obukhov (1941).

By performing a Buckingham Pi dimensional analysis, we can make only one dimensionless group from these three variables:

$$\pi_1 = \frac{S^3 \kappa^5}{\varepsilon^2}$$

We know that this Pi group must be equal to a constant, because there are no other Pi groups for it to be a function of.

Solving the above equation for S yields:

$$S(\kappa) = \alpha_k \varepsilon^{2/3} \kappa^{-5/3} \qquad (9.9.1)$$

where the α_k is known as the ***Kolmogorov constant***. The value of this constant has yet to be pinned down (Gossard, et.al., 1982), but it is in the range of $\alpha_k = 1.53$ to 1.68.

Homework

9.1 – Sara

9.2 – Cátia

9.4 – Diogo

9.6 – Florian

9.7 – Jason

9.9 – Maria

9.16 – Mariana