IC/95/267
INTERNAL REPORT
(Limited Distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LADDERS WITH TWO, THREE AND FOUR COUPLED ISING SPIN CHAINS. MAGNETIC PROPERTIES [1]

R. Mejdani[2], A. Gashi[3]
International Centre for Theoretical Physics, Trieste, Italy,

O. Ciftja[4]
International School for Advanced Studies (SISSA), Trieste, Italy

and

A. Lambros
Department of Physics, University of Thessaloniki, Thessaloniki, Greece.

MIRAMARE - TRIESTE
August 1995

[1]Submitted to Physica Status Solidi (b).
[2]Permanent address: Department of Physics, University of Tirana, Tirana, Albania.
[3]*On leave of absence from*: Department of Physics, University of Tirana, Tirana, Albania.
[4]E-mail: ciftja@neumann.sissa.it

ABSTRACT

We consider ladders with two, three and four coupled Ising spin chains, characterized by interchain and intrachain couplings, to study their magnetic behaviour for different ratios of interaction constants and different values of magnetic field, by using a transfer matrix method and a complex algorithm realising the diagonalization and the differentiation. It is interesting that when the competition between interchain and intrachain interactions is different from zero the magnetization goes from the value one to the value zero in a narrow interval of nonzero temperatures, where, also, the succeptibility exhibits a very high peak. This narrow interval, increasing the number of chains, is displaced towards the higher temperatures. Also, in the case of interchain antiferromagnetic couplings, it becomes clear the formation of interchain spin pairs.

The spin $-\frac{1}{2}$ ladder or the system of coupled spin chains may be realized in nature by vanadyl pyrophosphate $((VO)_2P_2O_7)$ or similar materials. All these intermediate systems are today important to gain further insight into the physics of one-dimensional spin chains and the two-dimensional high-T_c spin systems, both of which have shown interesting and unusnal magnetic and superconducting properties. It is plausible that experimental and theoretical studies of ladders may lead to other interesting physical phenomena.

Key words: Ising-like model. Magnetic properties. Transfer matrix method.

Classification numbers: *05.50, 75.10.Hk, 75.30.Cr*

1 Introduction

The recent discovery of two-dimensional Heisenberg spin systems in the copper- oxygen planes of the high temperature superconductors has further increased the interest in lower-dimensional Heisenberg ferromagnets. Today, to gain further insight into the physics of one-dimensional spin chains and the two-dimensional high- T_c spin systems, one may study systems which are intermediate between these two cases ([1]). So, the orthorhombic compound of vanadyl pyrophosphate $(VO)_2P_2O_7$ clearly shows a ladder configuration of $S = \frac{1}{2}$ V^{4+} ions in its crystal lattice ([2]). Also, several Cu^{2+} compounds are known from crystallographic studies at room temperature to form related one-dimensional spin ladder configurations. Concretely, two ladder systems: $Sr_{n-1}Cu_{n+1}O_{2n}$ series([3]-[4]) and $La_{4+4n}Cu_{8+2n}O_{14+8n}$ series([5]) have been investigated. In a parallel series of investigations the properties of spin ladders with ferromagnetic rung couplings (which are more closely related to spin -1 chains) have also been discussed([6]-[8]). Due to speculations that the $S = \frac{1}{2}$ square - lattice antiferromagnetic copper - oxygen planes might be a vital component of the mechanism of high- temperature superconductivity, many recent studies of antiferromagnetic Heisenberg spin ladders have subsequently appeared in the literature([9]-[20]).

In [21] we have considered the simplest case of a spin ladder, which is an Ising spin ladder, consisting of two chains of magnetic ions (or spins) coupled by an interaction of strength J_1 along the chains and J_2 between them, to study in details the behaviour of the specific heat. In the present work we shall consider ladders with two, three and four coupled Ising spin chains, characterized by (ferromagnetic and antiferromagnetic) interchain and (ferromagnetic) intrachain couplings, to study their magnetic properties as magnetization and suscepbtibility, for different ratios of interaction constants and different values of magnetic field, by using a transfer matrix method and a complex algorithm for diagonalization and differentiation.

2 Ladder with two coupled Ising spin chains

2.1 The model and the transfer matrix method.

The related low-dimensional system investigated in this section is an Ising "ladder" with N columns or rungs. This system is the standard spin Ising model on a ladder of two coupled spin chains, refered 1 and 2. The ladder Hamiltonian with a strength- J_1 interaction along the long (chain) axis of the ladder, and a J_2 interaction across the rungs, is given by:

$$H = -J_1 \underbrace{\sum_{i=1}^{N}}_{(chains)} (S_{1,i}S_{1,i+1} + S_{2,i}S_{2,i+1}) - J_2 \underbrace{\sum_{i=1}^{N}}_{(rungs)} S_{1,i}S_{2,i} - B\sum_{i=1}^{N}(S_{1,i} + S_{2,i}) \quad (1)$$

where B is the magnetic field and $S_{1,i}$ or $S_{2,i} = \pm 1$ at site $(1,i)$ or $(2,i)$ of the ladder. We will consider also ladders with periodic boundary conditions in the long axis direction.

Developping the Hamiltonian in the "elementary segments"(corresponding to the modified squares) of the type:

$$-[J_1(S_{1,i}S_{1,i+1} + S_{2,i}S_{2,i+1}) + \frac{J_2}{2}(S_{2,i}S_{1,i} + S_{1,i+1}S_{2,i+1}) + \frac{B}{2}(S_{2,i} + S_{1,i} + S_{1,i+1} + S_{2,i+1})] \quad (2)$$

the partition function can be written as:

$$Z = \sum_{S_{1,i},S_{2,i}} exp(-H/k_BT) = Tr(\mathbf{T}^{\mathbf{N}}) \tag{3}$$

where k_B is the Boltzmann constant, T - the absolute temperature and $\mathbf{T}$ - the transfer matrix.

Denoting $k_1 = (J_1/k_BT) = k = (1/t)$ (with $t = (k_BT/J_1)$ - the reduced temperature), $k_2 = (J_2/k_BT) = ek$ (with $e = (J_2/J_1)$ - the ratio of interaction constants), $h_o = (B/k_BT) = kh$ (with $h = (B/J_1)$ - the reduced field) and $x = exp(k)$, $y = exp(ek)$, $z = exp(kh)$ the transfer matrix $\mathbf{T}$, with 2^2 x 2^2 elements and symmetric related to the main diagonal, constructed following the decreasing order of bits numbers from the configuration (1111) (four spins $S_{1,i}, S_{1,i+1}, S_{2,i}, S_{2,i+1}$ are "up") to the configuration (0000) (four spins point "down"), is given by:

$$\mathbf{T} = \begin{pmatrix} x^2yz^2 & z & z & x^{-2}y \\ z & x^{-2}y^{-1} & x^2y^{-1} & z^{-1} \\ z & x^2y^{-1} & x^{-2}y^{-1} & z^{-1} \\ x^{-2}y & z^{-1} & z^{-1} & x^2yz^{-2} \end{pmatrix} \tag{4}$$

This matrix, in the limit of $h = 0$ (in absence of a magnetic field) and in the isotropic limit $e = 1$ (equal interchain and intrachain copulings) is equal to the matrix given in [22].

From the diagonalization of this matrix we find the larger eigenvalue λ_{max}. Since the maximum eigenvalue λ_{max} is equal to the grand partition function per "elementary segment" (or 2 spins), in the thermodynamic limit, the thermodynamic potential or the free energy per spin , in unity of J_1, is derived as:

$$f = -(k_BT/J_1)\lim_{N\to\infty}[(2N)^{-1}ln(\lambda_{max})^N] = -\frac{1}{2k}ln(\lambda_{max}) \tag{5}$$

From the thermodynamic relation, for the magnetization per spin m, we can write:

$$m = -\frac{\partial f}{\partial h} = \frac{1}{2k}\frac{\partial}{\partial h}ln(\lambda_{max}) \tag{6}$$

while for the susceptibility X we have:

$$X = \frac{\partial m}{\partial h} = \frac{1}{2k}\frac{\partial^2}{\partial h^2}ln(\lambda_{max}) \tag{7}$$

Based on the relations (4)-(7) we have constructed a complex algorithm , which realizes progressively the diagonalization, the selection of the larger eigenvalue and the first and second differentiation. Through this numerical procedure we shall study the behaviour of the magnetization and susceptibility related to temperature, interaction ratio and magnetic field.

2.2 The magnetization and the susceptibility

In the case $B = 0$ (or $h = 0$) following the procedure and the algorithm mentioned above we have studied the variation of the magnetization (per spin) m and the susceptibility X versus the reduced temperature t for different values of interaction ratio $\epsilon(\epsilon > 0)$. For illustration, in Fig.1a,b are represented this variations for some values of ϵ ($\epsilon = 0.1, 0.2, 1, 2$). It is clearly visible from these $(m - t)$ or $(X - t)$ diagrams that in a narrow critical region of the reduced temperature t the magnetization goes from the value one to the value zero (Fig.1a), while the susceptibility in this narrow interval of temperature has a very high peak of the order of magnitude $10^8 - 10^9$ (Fig.1b). For the $\epsilon = 0$ or in the $J_2 = 0$ -"chain limit", i.e., in the case of two chains without couplings across them, we find the same variation as in the case of a single chain in absence of the magnetic field. (In this case, diagonalizing the transfer matrix analitically for λ_{max} we obtain a value which is the square value of the eigenvalue for a single chain, i.e., based on (5), the same thermodynamic potential as in a single chain.)

Quite interesting is the case when $\epsilon < 0$ ($h = 0$). In this case, for $t = 0$, or in the ground state, the spins of each chain are parallel (inside one chain), but point in different directions ("up" and "down", or "down" and "up") from one chain to another. Controlling this case (even for very small values of ϵ, in the limit of our numerical accuracy) we have seen that the magnetization is zero in all region of temperatures. In Fig.2 is illustrated this quite special behaviour for $\epsilon = -0.01$ ($h = 0$). It means that a coupled invertion or a spin pairing across the chains is present, i.e., when a spin in one site of the rung is flipping up its partner in other site of the same rung is turning down or vice-versa. In the same figure we have represented, also, the variation of m versus t in the case of an antiferromagnetic interachain coupling with $\epsilon = -0.55$ and in the presence of a magnetic field $h = 0.5$.

In Fig.3a and Fig.3b, for illustration, are represented the $(m - t)$ and $(X - t)$ diagrams, respectively, in the presence of the magnetic field ($h = 0.5$ and $h = 1$), in the case of a ferromagnetic isotropic interchain coupling ($\epsilon = 1$).

3 Ladder with three coupled Ising spin chains

3.1 The model and the transfer matrix method.

We consider three coupled Ising spin chains. The interchain and intrachain interactions are taken as in section 2.1, i.e., J_1 is the interaction along the long (chain) axis of the triple ladder and J_2 -the interaction across the rungs. The Hamiltonian is given by:

$$H = -J_1 \underbrace{\sum_{i=1}^{N}}_{(chains)} (S_{1,i}S_{1,i+1} + S_{2,i}S_{2,i+1} + S_{3,i}S_{3,i+1}) - J_2 \underbrace{\sum_{i=1}^{N}}_{(rungs)} (S_{1,i}S_{2,i} + S_{2,i}S_{3,i}) -$$

$$- B\sum_{i=1}^{N}(S_{1,i} + S_{2,i} + S_{3,i}) \tag{8}$$

where B is the magnetic field and $S_{1,i}$, $S_{2,i}$ or $S_{3,i} = \pm 1$ at site $(1, i)$, $(2.i)$ or $(3, i)$ of the triple ladder. We will consider also ladders with periodic boundary conditions in the long axis direction.

Developping the Hamiltonian in the "elementary segments" :

$$-[J_1(S_{1,i}S_{1,i+1}+S_{2,i}S_{2,i+1}+S_{3,i}S_{3,i+1})+\frac{J_2}{2}(S_{3,i}S_{2,i}+S_{2,i}S_{1,i}+S_{1,i+1}S_{2,i+1}+S_{2,i+1}S_{3,i+1})+$$

$$+\frac{B}{2}(S_{3,i}+S_{2,i}+S_{1,i}+S_{1,i+1}+S_{2,i+1}+S_{3,i+1})] \tag{9}$$

the partition function can be written in a similar way as in (3):

$$Z=\sum_{S_{1,i},S_{2,i},S_{3,i}} exp(-H/k_BT)=Tr(\mathrm{T}^{\mathrm{N}}) \tag{10}$$

where T is now the (2^3 x 2^3)- transfer matrix, constructed by the same procedure as in 2.1.

Using the same notations as before, this matrix, with 64 nonindependent elements and symmetric related to the main diagonal, is given by:

$$\begin{pmatrix}
x^3y^2z^3 & xyz^2 & xz^2 & x^{-1}yz & xyz^2 & x^{-1}z & x^{-1}yz & x^{-3}y^2 \\
xyz^2 & x^{-1}z & x^{-1}y^{-1}z & x^{-3} & x^3z & xy^{-1} & x & x^{-1}yz^{-1} \\
xz^2 & x^{-1}y^{-1}z & x^3y^{-2}z & xy^{-1} & x^{-1}y^{-1}z & x^{-3}y^{-2} & xy^{-1} & x^{-1}z^{-1} \\
x^{-1}yz & x^{-3} & xy^{-1} & x^{-1}z^{-1} & x & x^{-1}y^{-1}z^{-1} & x^3z^{-1} & xyz^{-2} \\
xyz^2 & x^3z & x^{-1}y^{-1}z & x & x^{-1}z & xy^{-1} & x^{-3} & x^{-1}yz^{-1} \\
x^{-1}z & xy^{-1} & x^{-3}y^{-2} & x^{-1}y^{-1}z^{-1} & xy^{-1} & x^3y^{-2}z^{-1} & x^{-1}y^{-1}z^{-1} & xz^{-2} \\
x^{-1}yz & x & xy^{-1} & x^3z^{-1} & x^{-3} & x^{-1}y^{-1}z^{-1} & x^{-1}z^{-1} & xyz^{-2} \\
x^{-3}y^2 & x^{-1}yz^{-1} & x^{-1}z^{-1} & xyz^{-2} & x^{-1}yz^{-1} & xz^{-2} & xyz^{-2} & x^3y^2z^{-3}
\end{pmatrix} \tag{11}$$

From the diagonalization of this matrix we find the larger eigenvalue λ_{max} or the grand partition function per "elementary segment" (or 3 spins). In the thermodynamic limit, the thermodynamic potential or the free energy f, in unity of J_1, the magnetization m and the susceptibility X, per spin, are derived, respectively, as:

$$f=-\frac{1}{3k}ln(\lambda_{max}), m=\frac{1}{3k}\frac{\partial}{\partial h}ln(\lambda_{max}), X=\frac{1}{3k}\frac{\partial^2}{\partial h^2}ln(\lambda_{max}) \tag{12}$$

Based on these relations, using the same numerical procedure as in the section 2 we study for the magnetization and susceptibility their behaviour related to the temperature, the interaction ratio and the magnetic field.

3.2 The magnetization and the susceptibility

In the case $h=0$ or $h\neq 0$ following the procedure and the modified algorithm mentioned above, we have studied the variation of the magnetization (per spin) m and the susceptibility X versus the reduced temperature t for different values of interaction ratio $e(e>0)$. Also, in theses cases we obtain the same behaviour for $(m-t)$ or $(X-t)$ diagrams as in the section 2.2, but the narrow critical interval is displaced to higher values of temperatures. For the $e=0$ or in the $J_2=0$ -"chain limit" and $h=0$, i.e., in the case of three spin chains without couplings across them and with $h=0$, we find the same variation as in the case of a single chain in absence of the magnetic field. (In this case, diagonalizing the transfer matrix (11) analitically for λ_{max} we obtain a value which is the cubic value of the eigenvalue for a single chain, i.e., based on (12), we obtain the same thermodynamic potential as in a single chain.)

Very different is the case when $e < 0$ ($h = 0$), in which, for $t = 0$, the spins of each chain are parallel (inside one chain), but point in different directions ("up" and "down", or "down" and "up") from one chain to another. Controlling this case (even for very small values of e, in the limit of our numerical accuracy) we have seen that $m \to \frac{1}{3}$ when $t \to 0$. In Fig.4 is illustrated this quite interesting behaviour for $e = -0.001, e = -0.01$ and $e = -0.1$ ($h = 0$). It means that a coupled invertion or a **spin pairing** across only two chains (1 and 2 or 2 and 3) is present, i.e., when a spin in one site of the rung is flipping up its partner in other nearest neighbour site of the same rung is turning down or vice-versa. The third chain behaves like independent. Thus, in **three spins**, situated in one rung of the three chains, only **one spin** contributes to the magnetization m in $t = 0$.

4 Ladder with four coupled Ising spin chains

4.1 The model and the transfer matrix method.

We consider now four coupled Ising spin chains (1, 2,3 and 4). The intrachain interactions and the interchain (between the 1-st and 2-nd chains and the 3-rd and 4-th chains) interactions are taken as in section 2.1, i.e., J_1 is the interaction along the long (chain) axis of the quadruple ladder and J_2 -the interaction across the rungs 1-2 and 3-4. The interchain interaction between the 2-nd and the 3-rd chain is taken J_o.The Hamiltonian is given by:

$$H = -J_1 \underbrace{\sum_{i=1}^{N}}_{(chains)} (S_{1,i}S_{1,i+1} + S_{2,i}S_{2,i+1} + S_{3,i}S_{3,i+1} + S_{4,i}S_{4,i+1}) - J_2 \underbrace{\sum_{i=1}^{N}}_{(rungs)} (S_{1,i}S_{2,i} + S_{3,i}S_{4,i})$$

$$- J_o \underbrace{\sum_{i=1}^{N}}_{(rungs)} S_{2,i}S_{3,i} - B\sum_{i=1}^{N}(S_{1,i} + S_{2,i} + S_{3,i} + S_{4,i}) \qquad (13)$$

where B is the magnetic field and $S_{1,i}$, $S_{2,i}$, $S_{3,i}$ or $S_{4,i} = \pm 1$ at site $(1,i)$, $(2,i)$, $(3,i)$ or $(4,i)$ of the double ladder. We will consider also ladders with periodic boundary conditions in the long axis direction.

Developping the Hamiltonian in the "elementary segments" :

$$-[J_1(S_{1,i}S_{1,i+1}+S_{2,i}S_{2,i+1}+S_{3,i}S_{3,i+1}+S_{4,i}S_{4,i+1})+\frac{J_2}{2}(S_{4,i}S_{3,i}+S_{2,i}S_{1,i}+S_{1,i+1}S_{2,i+1}+S_{3,i+1}S_{4,i+1})+$$

$$+\frac{J_o}{2}(S_{3,i}S_{2,i}+S_{2,i+1}S_{3,i+1})+\frac{B}{2}(S_{4,i}+S_{3,i}+S_{2,i}+S_{1,i}+S_{1,i+1}+S_{2,i+1}+S_{3,i+1}+S_{4,i+1})] \quad (14)$$

the partition function can be written in a similar way as in (3) or (10):

$$Z = \sum_{S_{1,i},S_{2,i},S_{3,i},S_{4,i}} exp(-H/k_BT) = Tr(\mathrm{T}^{\mathrm{N}}) \qquad (15)$$

where **T** is now the (2^4 x 2^4)- transfer matrix, constructed by the same procedure as in 2.1. Using the same notations as before, adding $e_o = (J_o/J_1)$ and $u = exp(J_o/k_BT) = exp(e_ok)$, this matrix, with 256 nonindependent elements and symmetric related to the main diagonal, is expressed through x, y, z and u. We will not give here its expression.

From the diagonalization of this matrix we find the larger eigenvalue λ_{max} or the grand partition function per "elementary segment" (or 4 spins). In the thermodynamic limit, the thermodynamic potential or the free energy f, in unity of J_1, the magnetization m and the susceptibility X, per spin, are derived, respectively, as:

$$f = -\frac{1}{4k}ln(\lambda_{max}), m = \frac{1}{4k}\frac{\partial}{\partial h}ln(\lambda_{max}), X = \frac{1}{4k}\frac{\partial^2}{\partial h^2}ln(\lambda_{max}) \qquad (16)$$

Based on these relations, using the same numerical procedure as in the section 2 and 3 we study the behaviour of the magnetization and susceptibility related to the temperature, the interaction ratio and the magnetic field.

4.2 The magnetization and the susceptibility

Following the procedure and the modified algorithm mentioned above, we have studied the variation of the magnetization (per spin) m and the susceptibility X versus the reduced temperature t for different values of interaction ratios e and e_o. For $h = 0$ and $e > 0$, $e_o > 0$ we obtain a similar behaviour for $(m - t)$ or $(X - t)$ diagrams as in the section 2.2 or 3.3, but the narrow critical interval is displaced to higher values of temperatures. This is illustrated in Fig.5a,b for the isotropic cases of three considered variants when $e = 1$ or $e = e_o = 1$ and $h = 0$. Also, we have seen that the influence of e (of J_2) is bigger than that of e_o (of J_o) in increasing these temperatures, which based on interaction configurations is quite normal. In the case $e_o = 0$ or $J_o = 0$ (and $e \neq 0$) the results for the ladder with four chains are reduced to them of a simple ladder (section 2.2), while in the case $e = 0$ or $J_2 = 0$ (and $e_o \neq 0$) the double ladder is reduced to a single ladder and two independent single chains. In the last case for $h = 0$ when $t \to 0$ ($t \neq 0$) the average value of magnetization per spin, considering all spin configurations (the magnetization m of single chains is equal to zero for $t \neq 0$) tends to (1/2), i.e., $m \to 0.5$, which is quite clear in Fig.6, but strictly at $t = 0$ there is a "transition" jump (the inset in Fig.6) to $m = 1$ (for single chain $m = 1$ at $t = 0$).

*Very different are the cases when one, or both of the interchain interactions are an*tiferromagnetic (and different from zero). For $t = 0$, the spins of each chain are parallel (inside one chain), but can point to different directions from one chain to another. There are three possible cases: a)$e > 0$ and $e_o < 0$; c)$e < 0$ and $e_o > 0$; a) $e < 0$ and $e_o < 0$. In all these cases the magnetization m is zero for $h = 0$, which can be explained easily by antiferromagnetic spin pairings.

5 Conclusions

In this paper we have considered Ising spin ladders with two, three or four chains to study in detail the behaviour of the magnetization and the susceptibility related to the different values of the temperature, the ratio of interaction constants and the magnetic field. An important feature for the system is the spin pairing when an antiferromagnetic interchain coupling is present. Also, increasing the number of chains from two to four we see clearly the displacement of the narrow critical regions towards the higher values of the temperature.

To understand better the behaviour of this kind of low-dimensional systems for an arbitrary, but limited number of chains, we are trying to construct a general algorithm to generate the elements of the transfer matrix and after to realize progressively the diagonalization, and the selection of the larger eigenvalue, and finally, the first and second differentiation. This will be the object of a future work.

Acknowledgements

Two of authors (R.M. and A.G.) would like to thank for its hospitality the International Centre for Theoretical Physics, Trieste, where the main part of this work was done.

References

[1]. R.S.Eccleston, T.Barnes, J.Brody, J.W.Johnson ; Phys. Rev. Lett., **73**, 2626 (1994).
[2]. D.C.Johnston, J.W.Johnson, D.P.Goshorn, A.J.Jacobson; Phys. Rev. B, **35**, 219 (1987).
[3]. M.Takano, Z.Hiroi, M.Azuma, Y.Takeda; Jpn. J. Appl. Phys., **7**, 3 (1992).
[4]. Z.Hiroi, M.Azuma, M.Takano, Y.Bando; J. Solid State Chem., **95**, 230 (1991).
[5]. R.J.Cava et al., J. Solid State Chem., **94**, 170 (1991).
[6]. K.Hida, J. Phys.Soc.Jpn.,**60**,1347 (1991).
[7]. S.Takada, H.Watanabe, J. Phys.Soc.Jpn., **61**, 39 (1992).
[8]. H.Watanabe, K.Nomura, S.Takada, J. Phys.Soc.Jpn., **62**, 2845 (1993).
[9]. E.Dagotto, J.Riera, D.Scalapino, Phys.Rev. B , **45**, 5744 *(1992).*
[10]. T.Barnes, E.Dagotto, J.Riera, E.S.Swanson, Phys.Rev. B, **47**, 3196 (1993).
[11]. J.Riera, Phys. Rev. B, **49**, 3629 (1994).
[12]. T.Barnes, J.Riera, Phys. Rev. B, **50**, 6817 (1994).
[13]. R.M.Noack, S.R.White, D.J.Scalapino, Phys. Rev. Lett., **73**, 882 (1994).
[14] S.R.White, R.M.Noack, D.J.Scalapino, Phys. Rev. Lett., **73**, 886 (1994).
[15]. M.Reigrotzki, H.Tsunetsugu, T.M.Rice, J. Phys.: Condens. Matter, **6**, 9235 (1994).
[16]. S.Gopalan, T.M.Rice, M.Sigrist, Phys. Rev. B, **49**, 8901 (1994).
[17]. M.Sigrist, T.M.Rice, F.C.Zhang, Phys. Rev. B, **49**, 12058 (1994).
[18]. H.Tsunetsugu, M.Troyer, T.M.Rice, Phys.Rev. B, **49**, 16078 (1994).
[19]. D.Poilblanc, H.Tsunetsugu, T.M.Rice, Phys.Rev. B, **50**, 6511 (1994).
[20]. M.Azzouz, L.Chen, S.Moukouri, Phys. Rev. B, **50**, 6233 (1994).
[21]. R.Mejdani, A.Lambros,*A ladder Ising spin configuration. Heat capacity*,**IC/95/178**, (ICTP,Trieste-Italy) 1995, p.11.
[22]. C.J.Thompson, *Mathematical Statistical Mechanics*, Princeton Univ.Press. New Jersey,pp 177-220, 1979.

Figure Captions

Fig.1a. The variation of m versus t for some values of e ($e = 0.1, 0.2, 1, 2$) and $h = 0$ (the simple ladder).
Fig.1b. The corresponding $(X - t)$ diagrams.

Fig.2 Two variations of m versus t in the case of antiferromagnetic interchain couplings for $e = -0.01$, $h = 0$ and $e = -0.55$, $h = 0.5$.

Fig.3a. The $(m - t)$ diagrams in the presence of a magnetic field ($h = 0.5$ and $h = 1$), in the case of a ferromagnetic isotropic interchain coupling ($e = 1$).
Fig.3b. The corresponding $(X - t)$ diagrams.

Fig.4. The variation of the magnetization m versus the temperature t for the ladder with three chains when $e = -0.001$, $e = -0.01$ and $e = -0.1$ ($h = 0$).

Fig.5. The displacement of the critical narrow region towards the higher values of the temperature, for isotropic interactions and $h = 0$, represented in:
a) $(m - t)$ diagrams
b) $(X - t)$ diagrams

Fig.6. The $(m - t)$ diagram for two extreme cases: $e = 0$, $e_o = 1$ and $e = 1$, e_o of the ladder with four chains ($h = 0$).

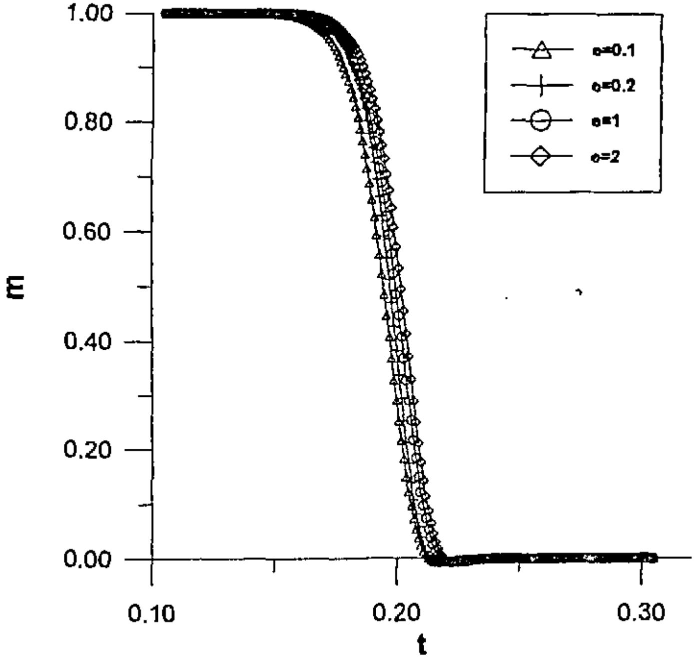

1.00
0.80
0.60
0.40
0.20
0.00
m
0.10
0.20
0.30
t
e=0.1
e=0.2
e=1
e=2

Fig. 1a

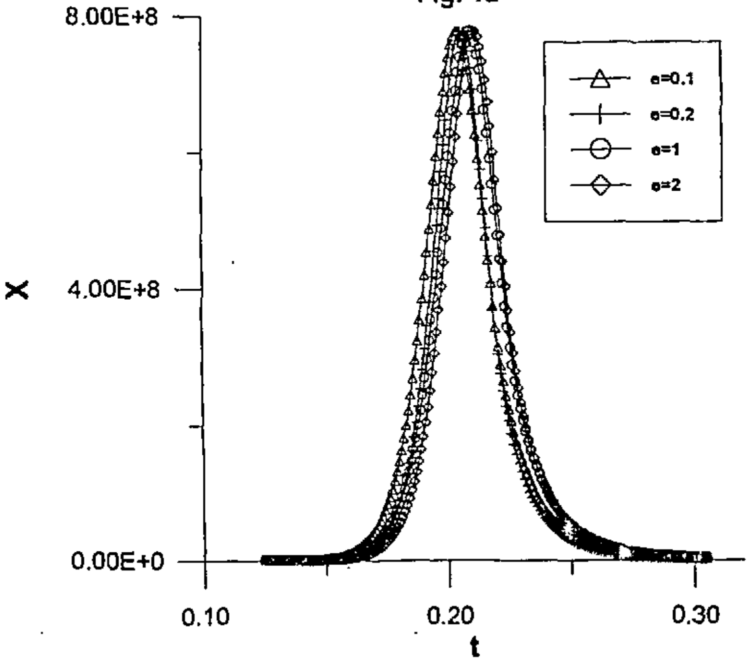

8.00E+8
4.00E+8
0.00E+0
X
0.10
0.20
0.30
t
e=0.1
e=0.2
e=1
e=2

Fig. 1b

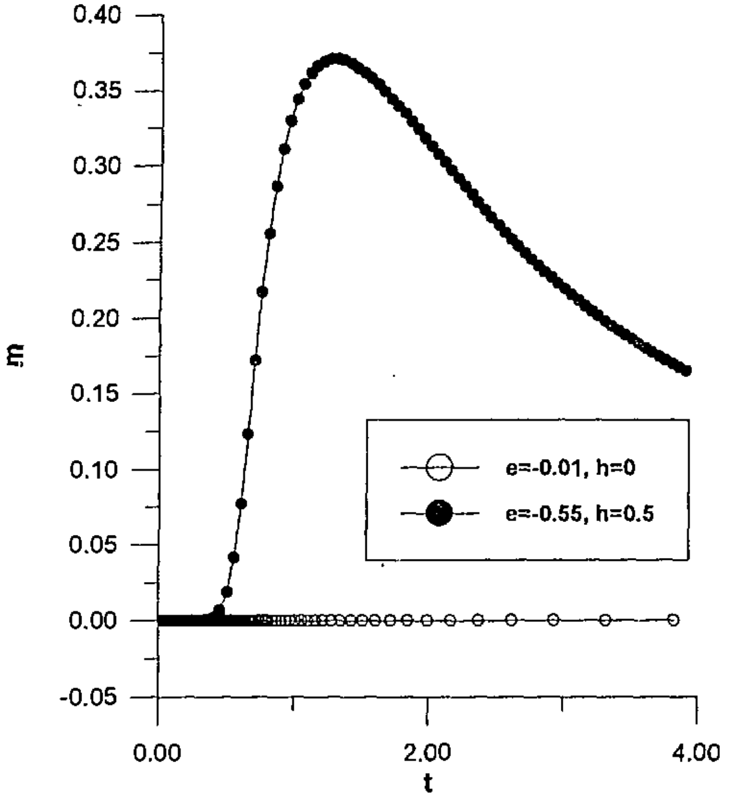

0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
m
0.00
2.00
4.00
t
e=-0.01, h=0
e=-0.55, h=0.5

Fig. 2

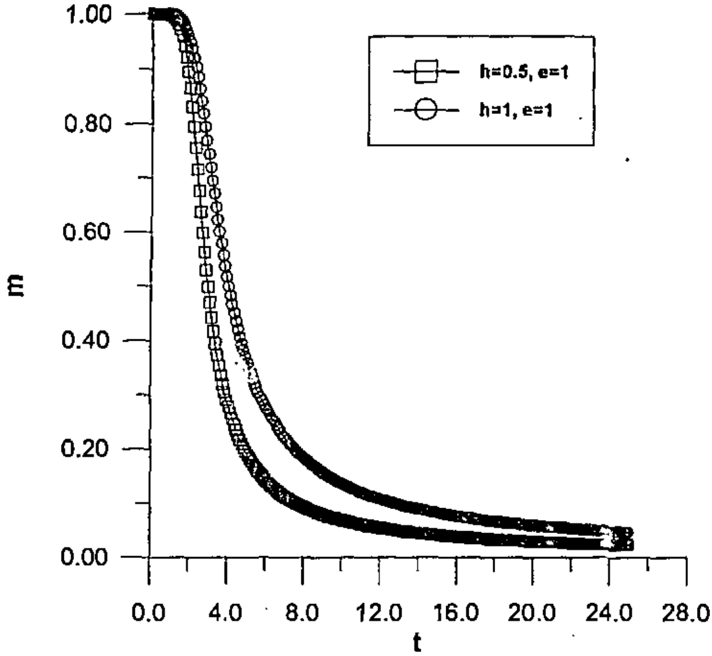

h=0.5, e=1
h=1, e=1
1.00
0.80
0.60
0.40
0.20
0.00
m
0.0
4.0
8.0
12.0
16.0
20.0
24.0
28.0
t

Fig. 3a

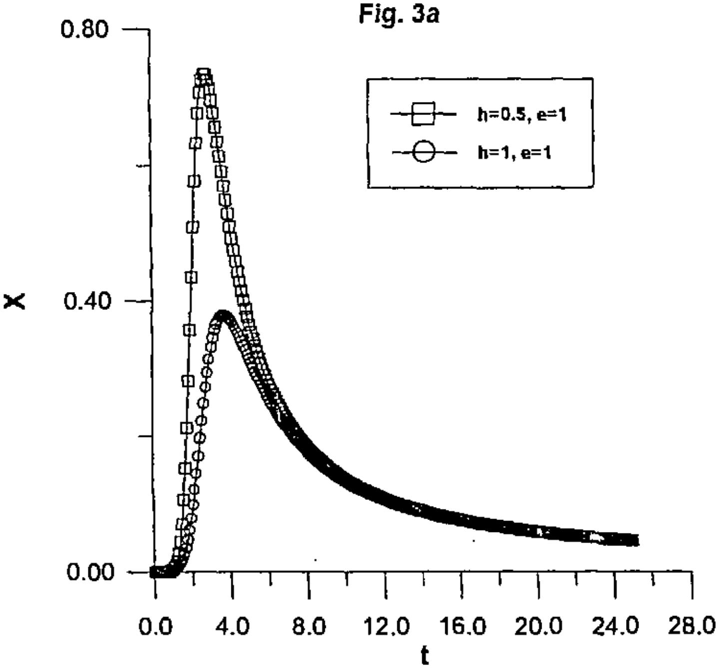

h=0.5, e=1
h=1, e=1
0.80
0.40
0.00
X
0.0
4.0
8.0
12.0
16.0
20.0
24.0
28.0
t

Fig. 3b

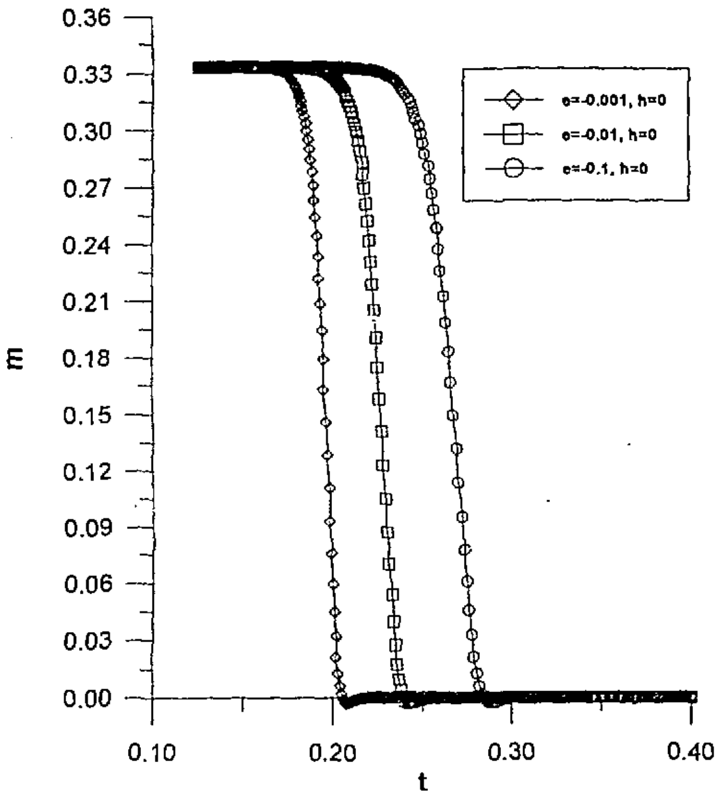
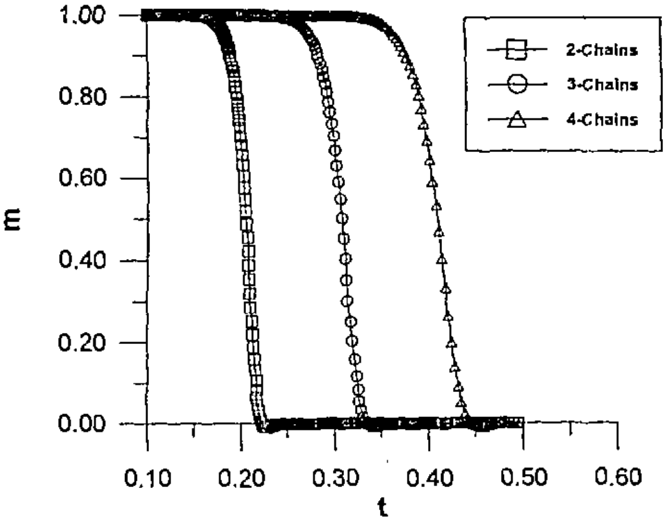
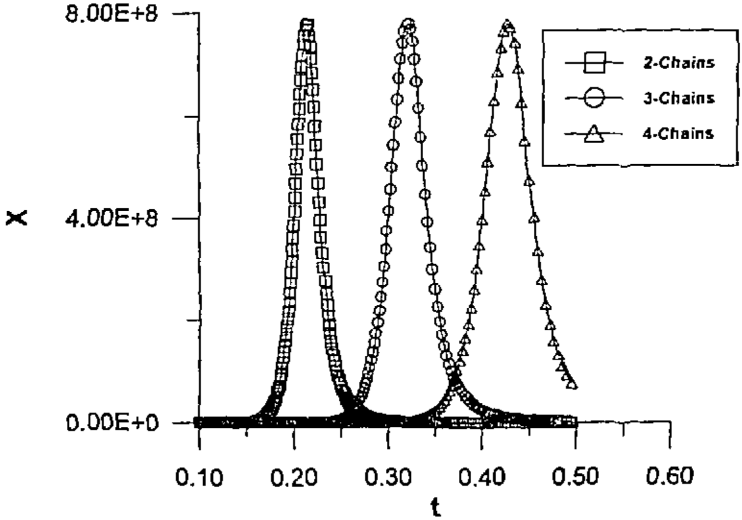

e=-0.001, h=0
e=-0.01, h=0
e=-0.1, h=0
m
0.36
0.33
0.30
0.27
0.24
0.21
0.18
0.15
0.12
0.09
0.06
0.03
0.00
0.10
0.20
0.30
0.40
t

Fig. 4


1.00
0.80
0.60
0.40
0.20
0.00
m
0.10
0.20
0.30
0.40
0.50
0.60
t
2-Chains
3-Chains
4-Chains
Fig. 5a


8.00E+8
4.00E+8
0.00E+0
X
0.10
0.20
0.30
0.40
0.50
0.60
t
2-Chains
3-Chains
4-Chains
Fig. 5b

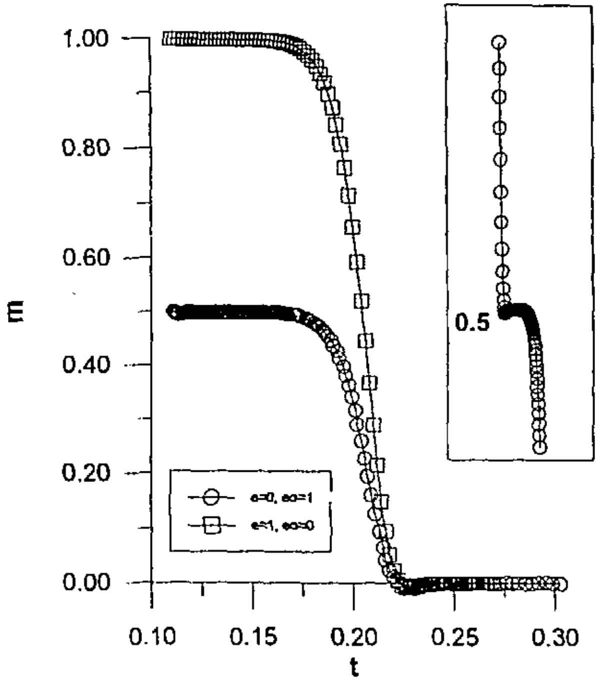


Fig. 6