(*) Please search and replace " ," by ", " throughout the text

Proceedings

Cosmological Perturbation Theory

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Abstract: In the paper, we have demonstrated the idea of primordial density fluctuation and the relation to the struture formation and observation. We start with the three fluid equation(Continuity Equation ,Eular Equation ,Poisson Equation) which govern the dynamic of particle in fluid and extend the idea on analyzing the density perturbation in the universe, we have derive the master equation of perturbation in 1) static universe without gravity ,2) static universe with gravity ,3) Expanding universe with gravity. For case 3, we had calculate the dark matter density contrast solutions in three different era, 1)Matter dominate era, 2)Radiation dominate era, 3) Dark energy dominate era, and have illustrate the different solutions in different conditions and its physical meaning.

Keywords: Newtonian Perturbation Theory, Expanding space ,Continuity Equation ,Eular Equation ,Poisson Equation ,Fourier transfrom;

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1. Introduction

The generally accepted theoretical framework for the formation of structure is the gravitational instability , it assumes the early universe to have been almost perfectly smooth, with the exception of tiny density deviations with respect to the global cosmic background density and the tiny velocity perturbations from the general Hubble expansion.The minor density deviations vary from location to location. At one place the density will be slightly higher than the average global density, while a few Megaparsecs further the density may have a slightly smaller value than on average. The observed fluctuations in the temperature of the cosmic microwave background radiation are a reflection of these density perturbations[1],this imply the primordial density perturbations have been in the order of 10^{-5} . It is believed that the density perturbations are the product of the processes in the very early Universe and correspond to quantum fluctuations which during the inflationary phase expanded to macroscopic proportions.Under the influence of the involved gravity perturbations in the beginning of the universe , the tiny local deviations from the average density of the Universe and also the corresponding deviations(peculiar velocity) from the global cosmic expansion velocity (the Hubble expansion) will start to grow.In a homogeneous Universe the gravitational force is the same everywhere, but in a universe with tiny density perturbations, then the perturbations will induce local differences in gravity. In a higher density region, the extra of matter will exert an attractive gravitational force larger than the average value.In a low density regions, the deficit in matter will lead to a weaker force. Because of this differences in gravitational force , it will not accelerate at the same extent in different location druing dark energy dominant the universe. So, during its early evolution an overdensity will experience a gradually stronger deceleration of its expansion velocity due to global Hubble expansion. And when the region has become sufficiently overdense, the mass of the fluctuation will have grown so much that its expansion may decouples from the Hubble expansion and start to contract. If the pressure forces are not sufficient to counteract the infall, more and more matter aggregate, ultimately this will turn into a full collapse to form a gravitationally bound object,the type of objects form are determined by its scale, mass and surroundings of the initial fluctuation, like galaxy or cluster .On the opposite tendency, in case of density depressions, the deceleration given by the gravity is not enough to overcome the global Hubble expansion, then matter will displace further and further and ultimately leading a void in the matter distribution.

2. Newtonian Perturbation Theory

2.1. Structure Formation: The linear regime

The linear Newtonian Perturbation Theory successfully describles the gravitationally evolving cosmological density and perturbation fields and is also an adequate description of general relativity on scales well inside the Hubble radius and for non-relativistic matter after the decoupling of radiation and matter at recombination. Furthermore, the linear theoretical predictions fail soon after gravity increases beyond a level, meaning the density perturbations on a small scale appear have a much higher amplitude than those on larger scales, small-scale perturbations being the first ones to become nonlinear and develop into cosmic objects. Furthermore, the linear analysis of structure evolution is still valid for scales ~~at least in~~ larger than Megaparsec scales at the present cosmic epoch.

what conditions?

Please noted that in the last part of the project, in the section of radiation domination era, we have assumed certain conditions that have already proven in relativistic perturbation theory, so we can still use the linear pertubation theory in this period. And we will talk about it later.

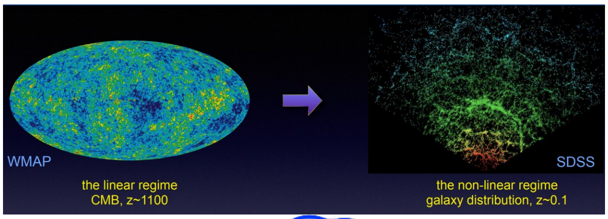


Figure 1. ...... caption?

(*) "show" is not a "prove"

Observations of the CMB on the left Fig.1 show that the perturbations at recombination (matter domination epoch) are still very much in the linear regime $|\delta| < 10^{-5}$, reflecting underlying primordial density and velocity perturbations. These are the seeds of the structure observed in the present Universe on the left of fig. 1. However, at the present-day the Universe has clearly entered the non-linear regime, and unless if the scale is larger than a few Mpc, the Universe is still in linear regime. Let us focus on the evolution of the density field in the $\rho(t)$ linear regime, which means that $|\delta| << 1$.

right

not yet defined

3. Perturbed Fluid Equations

Consider a non-relativistic fluid with mass density ρ, $P \ll \rho$ and velocity **u**. Denote the position vector of a fluid element by **r** and time by t. The equations of motion are given by basic fluid dynamics. Mass conservation implies the continuity equation. Let us first explain the continuity equation in the following graph.

3.1. Continuity Equation

In fluid mechanics, the equation for balancing mass flows and the associated change in density (conservation of mass) is called the continuity equation. Consider a very small control volume $\Delta x \Delta y \Delta z$,we will first consider only a one-dimensional compressible fluid flows through this volume in the x-direction. If the flow enters the volume element with a velocity $v_{x,in}$, it travels the infinitesimal distance $dx_{x,in} = v_{x,in}dt$ within the time dt. So we have a volume dV_x in flows into the control volume.

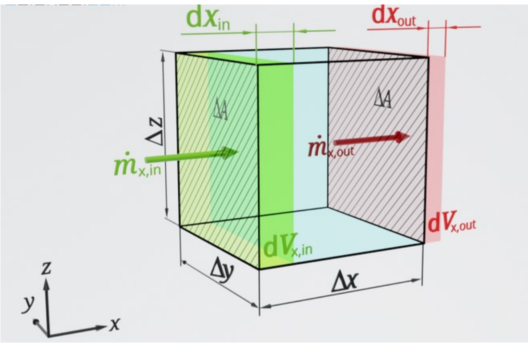


Figure 2 caption

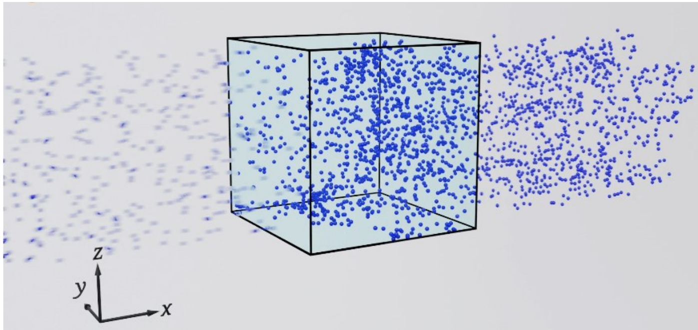


Figure 3 caption

$$dV_{x,in} = \Delta A \cdot dx_{x,in} = \Delta A \cdot v_{x,in} \cdot dt \tag{1}$$

you don't need the • in formulas!

If the density of the fluid is $\rho_{x,in}$, in at the point of entry into the volume element, the inflow mass dm_x is :

$$dm_{x,in} = dV_{x,in} \cdot \rho_{x,in} = \Delta A \cdot \rho_{x,in} \cdot v_{x,in} \cdot dt \tag{2}$$

Also the mass flux which is the mass flowing in the flow direction per unit time and unit area:

Q1

$$\dot{m}^*_{x,in} = \frac{dm_{x,in}}{\Delta A \cdot dt} = \rho_{x,in} \cdot v_{x,in} \quad \text{and similarly} \quad \dot{m}^*_{x,out} = \rho_{x,out} \cdot v_{x,out} \tag{3}$$

If more mass flows into the control volume than flows out, the mass inside increases. The rate of change of mass $\dot{m}$ in the control volume results from the difference of the mass flowing in and out .

$$\underbrace{\dot{m}}_{\text{change of mass inside the CV}} = \underbrace{\dot{m}_{x,in}}_{\text{inflowing mass into the CV}} - \underbrace{\dot{m}_{x,out}}_{\text{outflowing mass from the CV}} \tag{4}$$

italic

where the change in mass in dt is equal to mass flux time the infinitesimal area $\dot{m} = \dot{m}^* \cdot \Delta A$

\, or \quad

$$\dot{m} = \dot{m}^*_{x,in} \cdot \Delta A - \dot{m}^*_{x,out} \cdot \Delta A \quad \text{where} \Delta A = \Delta y \cdot \Delta z \tag{5}$$

with

The above equation can be shown in fig 3,(steady flow) a faster inflow in the left side and a slower outflow in the right side leading the accumulation of particle inside the volume therefore volume mass and density increase ~~by~~ the time.

Since mass flux is ρv , we can define a gradient of mass flux: $\frac{\partial \rho v_x}{\partial x}$. Therefore the change in mass flux $d\dot{m}^*_x$ in x direction is:

$$d\dot{m}^*_x = \frac{\partial \rho v_x}{\partial x} \cdot dx \tag{6}$$

Then the outflow of the mass flux of the control volume is [fig 2] (see Fig 2)

$$\dot{m}^*_{x,out} = \dot{m}^*_{x,in} + d\dot{m}^*_x = \dot{m}^*_{x,in} + \frac{\partial(\rho v_x)}{\partial x} \cdot dx \tag{7}$$

Then we substitate u this relation to equation 4,we obtain the temporal change of mass $\dot{m}$ inside the control volume:

$$\dot{m} = \dot{m}^*_{x,in} \cdot \Delta y \cdot \Delta z - (\dot{m}^*_{x,in} + \frac{\partial(\rho v_x)}{\partial x}) \cdot dx \cdot \Delta y \cdot \Delta z \tag{8}$$

$$\dot{m} = -\frac{\partial(\rho v_x)}{\partial x} \cdot dV \tag{9}$$

The negative sign indicates that if a positive gradient of mass flux in the back surface will lead to a decrease of mass inside the volume as the outflow is larger than the inflow, the volume mass will decrease.The equation 9 can be illustrated by fig.4 . In a incompressible fluid the velocity,the crossection area are different in the blue regions,but the density

explain better

Cross section

remain the same and also the volume is the same when the flow is within a time duration of dt, the flow in the narrower region will be slower.But in the case of compressible gas, the densities of different regions are different.

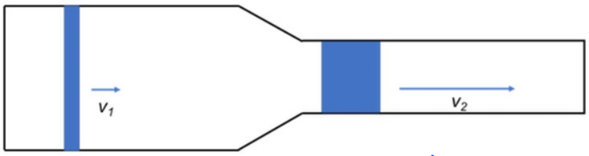


Figure 4 caption

So if the mass in the volume element generally changes over time, then its density ρ also changes over time:

$$\dot{m} = \frac{\partial(\rho)}{\partial t} \cdot dV \tag{10}$$

Then we can combine ing Eq. eq 9 and 10,we have the continuity equation for one-dimensional flow and similarly we can extend to three-dimensional s. The divergence of vector field of mass flux ρv:

$$\frac{\partial(\rho)}{\partial t} = -\frac{\partial(\rho v_x)}{\partial x} \quad (1D) \rightarrow \frac{\partial(\rho)}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho v) \quad (3D) \tag{11}$$

3.2. Euler Equation

The Euler equation describles the change in velocity of a fluid particle to the presence of a force and is regarded as a consequence of the conservation of momentum. Let consider an infinitesimal fluid volume dV with mass dm.We describe the motion of the fluid element from a fixed coordinate system (so-called Eulerian approach),and consider fluid element moves along an arbitrarily oriented streamline.

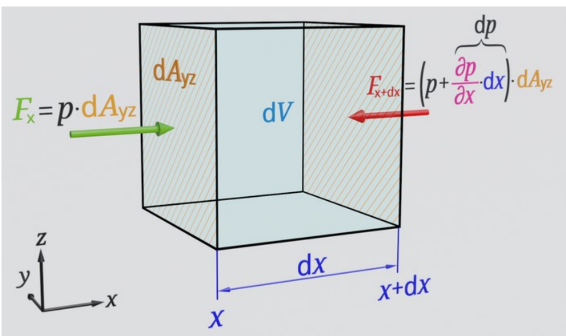


Figure 5 caption

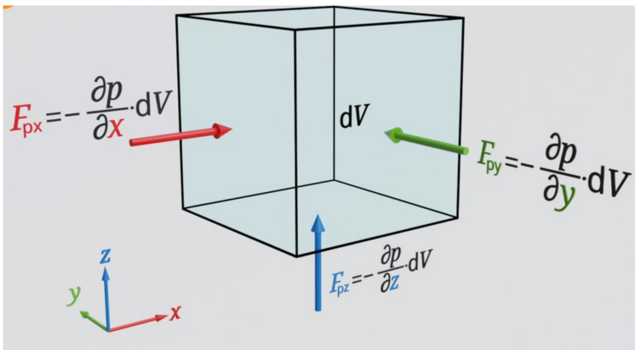


Figure 6 caption

3.3. Pressure forces on a fluid element

First consider the motion of the fluid element only in x-direction only. At a position x a pressure p is present. At this location the force Fx acts on the surface dA_{yz} of the fluid element.

$$F_x = p \cdot dA_{yz} \tag{12}$$

The pressure is not constant in a flow, but changes locally. This is because pressure differences are ultimately the reason why a flow comes from.Therefore, the pressure in the x-direction will change. Associate with a pressure gradient $\frac{\partial p}{\partial x}$, and the pressure change is $dp_x = \frac{\partial p}{\partial x} \cdot dx$.Then the force in position it x+dx is $F_{x+dx} = (p + \frac{\partial p}{\partial x} \cdot dx) \cdot dA_{yz}$ Since the force F_x and F_{x+dx} acting on surface dA_{yx} is in opposite directions,this give the resultant pressure force F_{px} in x direction is

$$F_{px} = F_x - F_{x+dx} = p \cdot dA_{yz} - (p + \frac{\partial p}{\partial x} \cdot dx) \cdot dA_{yz} = -\frac{\partial p}{\partial x} \cdot dx \cdot dA_{yz} = -\frac{\partial p}{\partial x} \cdot dV \tag{13}$$

If a positive pressure gradient $\frac{\partial p}{\partial x} > 0$,the force F_{x+dx} is larger than F_x,then the fluid element would would be slowed down in the positive x direction.Similarily in y and z direction.And we express the pressure force as vector notation:

$$\boldsymbol{F_p} = \begin{pmatrix} F_{px} \\ F_{py} \\ F_{pz} \end{pmatrix} = - \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix} \cdot dV == -\boldsymbol{\nabla} p \cdot dV \tag{14}$$

As pressure is a scalar field,applying the∇ operator to a scalar field,results in a vector field which point to the greatest increase of the scalar quantity (greatest pressure increase) .

simply assume ?

3.3.1. Shear forces and field forces on a fluid element

There are internal frictional forces occur, which are the greater the more viscous the fluid is, but in our case,we simpla friction-free flow, and the shear forces vanish.Also, other forces that act on the fluid element but cannot be neglected in general are field forces such as those caused by gravity. For example, there are (electrically charged particles/ferromagnetic fluid particles) in a fluid are possible, so that the flow is then influenced by an (external electric field/external magnetic field). Therefore the (resultant force/accelerating force) are contribute by $\boldsymbol{F_p}$ pressure force and $\boldsymbol{F_g}$ field forces .$\boldsymbol{F_a} = \boldsymbol{F_p} + \boldsymbol{F_g} = \boldsymbol{F_g} - \boldsymbol{\nabla} p \cdot dV$.Then we can obtain the material/substantial acceleration $\boldsymbol{a_{sub}}$ by dividing the $\boldsymbol{F_a}$ by the mass dm.

$$\boldsymbol{a_{sub}} = \frac{\boldsymbol{F_g} - \boldsymbol{\nabla} p \cdot dV}{dm} = \frac{\boldsymbol{F_g}}{dm} - \frac{\boldsymbol{\nabla} p \cdot dV}{\rho \cdot dV} \rightarrow \boldsymbol{a_{sub}} = \boldsymbol{g} - \frac{\boldsymbol{\nabla} p}{\rho} \quad \text{where} \quad dm = \rho \cdot dV \tag{15}$$

where the first term is gravitational acceleration, the second is pressure acceleration.

3.3.2. Temporal and convective acceleration

be understood as a consequence of 2 effects ?

The substantial acceleration can also interpret as two causes.A temporal accelerationa_{temp} and convective acceleration a_{con}. A fluid element viewed at a fix location changes its speed and direction by time in a unsteady flow, this refer to temporal acceleration.

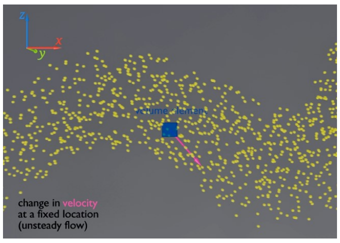


Figure 7. Temporal acceleration: Velocity 1 at fix position

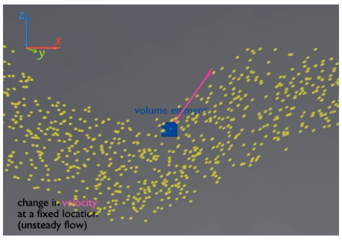


Figure 8. Temporal acceleration: Velocity 2 at fix position

The convective acceleration is due to the flow velocity changing from place to place.Also refer to fig. 4 . Even with steady flow, the velocity will be higher in narrower region than in a wider region.

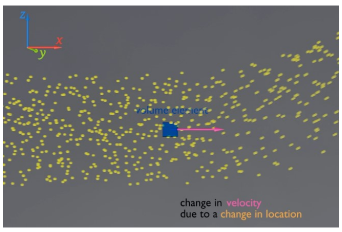


Figure 9. Convective acceleration:Change in velocity 1 due to change in position

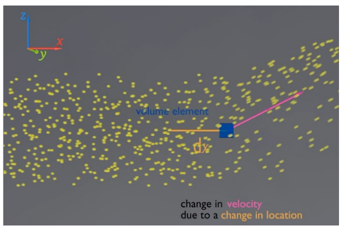


Figure 10. Temporal acceleration: Change in velocity 2 due to change in position

Let consider a fluid particle that move along a streamline s in 1 d, and the velocity is v .The substanial/total change in velocity is obtain from both the temporal change in velocity ∂vt within a time dt and the spatial change of velocity ∂vs gradient within the distance ds.We have

$$\underbrace{dv}_{\text{substantial change}} = \underbrace{\frac{\partial v}{\partial t}dt}_{\text{temporal change}} + \underbrace{\frac{\partial v}{\partial s}ds}_{\text{convective change}} \rightarrow a = \frac{\partial v}{\partial t}\frac{dt}{dt} + \frac{\partial v}{\partial s}\frac{ds}{dt} \tag{16}$$

$$a_{sub} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s}v \tag{17}$$

Therefore during 3 dimensional case,the equation become

$$\boldsymbol{a_{sub}} = \begin{pmatrix} \frac{\partial v_x}{\partial t} \\ \frac{\partial v_y}{\partial t} \\ \frac{\partial v_x}{\partial t} \end{pmatrix} + \begin{pmatrix} \frac{v_x \partial}{\partial x} & \frac{v_y \partial}{\partial y} & \frac{v_z \partial}{\partial z} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{18}$$

Now we equal both 15 and 18, yield

you didn't yet introduced The Poisson equation Say that is in section 3.4

$$D_t v = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{g} \rightarrow (\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla_r})\mathbf{v} = -\frac{\boldsymbol{\nabla_r} P}{\rho} - \boldsymbol{\nabla_r}\Phi \quad \text{(Possion equation)} \tag{19}$$

where $D_t = \partial_t + \mathbf{v} \cdot \boldsymbol{\nabla_r}$ is the Lagrangian/convective derivative, which means the derivative wrt a moving fluid element (as opposed to the Eulerian derivatives wrt some fixed grid point...).

3.4. Poisson equation

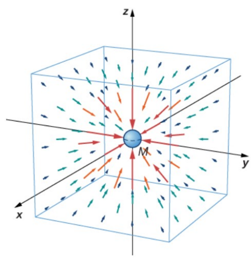
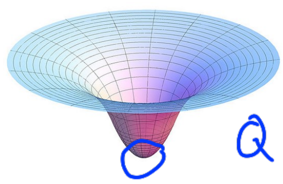


Figure 11. A three-dimensional representation of the gravitational field created by mass M



Q $\vec{x} \to \vec{0}$

Figure 12. Plot of a two-dimensional slice of the gravitational potential in and around a uniform spherical body.

Let us start with gravitational field. A gravitational field g[11] is a vector field that around a single particle of mass M, and every point of a vector pointing directly towards the particle.The force fields g at each point in space can also be express in term of Φ which is the scalar gravitational potential energy [12] per unit mass. That is the work per unit mass that are needed to move an object to a position.The gravitational field is defined using Newton's law of universal gravitation.

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = \frac{\boldsymbol{F}}{m} = -GM\frac{\boldsymbol{R}}{|\boldsymbol{R}|^2} \tag{20}$$

where Fis the gravitational force pointing to the mass M, m is the (test particle) that being attracted, R is the position of the test particle which a negtive sign is added as the force is oppose to the test particle position vector.

Then from gauss law of gravity state that the gravitational flux through any closed surface is proportional to the enclosed mass density, this give the diffential form:

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G\rho \tag{21}$$

Combine both 20 and 21:

$$\nabla_r^2 \Phi = 4\pi G\rho \tag{22}$$

{\it plus}

Now, we have the three main equations that to govern the dynamics,we introduce a small perturbations into the homogeneous background. We seperate the density,pressure, velocity and gravitational potential into background values + perturbations.For instance $\rho(t,r) = \overline{\rho(t)} + \delta\rho(t,r)$. Let us take our progress step by step from case 1 :Static space without gravity, case 2 :Static space with gravity and case 3: Expanding space.Also we assume the fluctuations are small, so we can simplify the problem by linearization . Keep in remind linearization means that when $\delta\rho$ and δv...are small, we can neglect all higher order terms (those with $(\delta\rho)^2$, $(\delta v)^2$, or $\delta\rho\delta v$ .....). From now on,we change the notation of velocity from v to u.

4. Case 1: Static space without gravity

Condition: Consider absence of gravity (potential $\Phi = 0$) and the space without expansion,background velocity $\overline{\boldsymbol{u}} = 0$, the latter implies the density ~~compose of~~ background $\overline{\rho}$ ~~as~~ is constant and small perturbations depend on the position and time $\delta\rho(\boldsymbol{r},t)$. $\rho(\boldsymbol{r},t) = \overline{\rho} + \delta\rho(\boldsymbol{r},t)$. The continuity equation becomes:

$$\partial_t(\overline{\rho} + \delta\rho(\boldsymbol{r},t)) + \boldsymbol{\nabla_r} \cdot (\overline{\rho} + \delta\rho(\boldsymbol{r},t))(\overline{\boldsymbol{u}} + \boldsymbol{\delta u}) = 0 \tag{23}$$

inside the parenthesis are

The first terms ~~on the left is~~ zero as ~~it is~~ constant.

After linearizartion which drops the they are second order term $\boldsymbol{\nabla_r} \cdot \delta\rho(\boldsymbol{r},t)\boldsymbol{\delta u}$,we have

$$\partial_t\delta\rho(\boldsymbol{r},t) = -\overline{\rho}\boldsymbol{\nabla_r} \cdot \boldsymbol{\delta u} \tag{24}$$

perturbing the with we have:

Similarily ~~from~~ Euler equation, with $\boldsymbol{\nabla_r}\Phi = 0$, ~~substitute the pertubations~~ (check if re-wording is OK)

$$[\partial_t + (0 + \boldsymbol{\delta u}) \cdot \boldsymbol{\nabla_r}](0 + \boldsymbol{\delta u}) = \frac{-\boldsymbol{\nabla_r}(\overline{P} + \delta P(\boldsymbol{r},t))}{\overline{\rho} + \delta\rho(\boldsymbol{r},t)} \tag{25}$$

% Rearranging the density to left side,then linearizing, the term $[(\boldsymbol{\delta u}) \cdot \boldsymbol{\nabla_r}](\boldsymbol{\delta u})(\overline{\rho} + \delta\rho(\boldsymbol{r},t))$ vanishes and ~~is then vanish~~,on the right side $\boldsymbol{\nabla_r}\overline{P} = 0$. So:

$$\partial_t(\overline{\rho} + \delta\rho(\boldsymbol{r},t))(\boldsymbol{\delta u}) = -\boldsymbol{\nabla_r}\delta P. \tag{26}$$

Linearizing again, the second order term vanishes:

$$\overline{\rho}\partial_t\boldsymbol{\delta u} = -\boldsymbol{\nabla_r}\delta P \tag{27}$$

And now we can combine the two equation by taking ∂_t in Eq. 24 and $\boldsymbol{\nabla_r}\cdot$ in Eq. 27 and we derive the (P.D.E) wave equation

$$\partial_t^2\delta\rho - \nabla_r^2\delta P = 0 \tag{28}$$

In our case, the fluctuations are called Adiabatic fluctuationsin which the pressure fluctuations are proportional to the density fluctuations, $\delta P = c_s^2\delta\rho$,where c_s is the adiabatic speed of sound.

$$[\partial_t^2 - c_s^2\nabla_r^2]\delta\rho = 0 \tag{29}$$

say by words the first time you use acronyms

In order to simplify the problem, it is more easy to treat a O.D.E problem than a P.D.E problem, that we can decompose into different modes by fourier transform. As perturbed density field can be written as sum of plane waves $\delta\rho = Ce^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ of different wave numbers/mode k. As $\delta\rho$ is a scaler, this indicate C is also a scaler magnitude of the fluctuations and $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$, this formula can interpret as in each position in space $\boldsymbol{k}\cdot\boldsymbol{r}$, ~~they have their own~~ oscillation $e^{i\omega t}$, and that oscillations are according to $\omega = c_s k$, and propagrate in $\boldsymbol{k}$ direction.

there is a time

Shows that

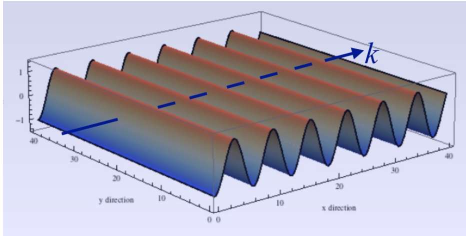


Figure 13. K th mode wave vector

The fourier transform:

$$\text{FT: } F[\delta\rho_{\boldsymbol{r}}(\boldsymbol{t})] \doteq \delta(\widehat{\boldsymbol{k},t}) = \int d^3 r e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta\rho_{\boldsymbol{r}}(\boldsymbol{t}) \quad \rightarrow \quad \delta(\boldsymbol{r},t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\widehat{\delta\rho_{\boldsymbol{k}}(\boldsymbol{t})} \tag{30}$$

Q2 Therefore we obtain each ~~of~~ ~~the~~ mode $\delta(\boldsymbol{k},t)$ from FT$[\delta(\boldsymbol{r},t)]$,then we have the solution in Eq. 32 and substitute/written as a form of superposition of all modes (on the right side of Eq. 30). Also in fourier space , $\nabla \rightarrow i\boldsymbol{k}$,therefore $\nabla^2 \rightarrow -k^2$, which is the eigenvalue,this scalar simplifer the problem to homogeneous Linear ODE of the form y''+ay'+by =0. Q2

$$[\partial_t^2 + c_s^2 k^2]\delta\rho_{\boldsymbol{k}} = 0 \tag{31}$$

Solving ODE: As $y = e^{\lambda t}$ is the solution of it. Substituting ~~become~~ gives $(\lambda^2 + a\lambda + b)e^{\lambda t} = 0$, and we check the characteristic eqaution. In our case Eq.31 a =0, $b = c_s^2 k^2$ We find that in our case ~~is~~ there are complex conjugate roots $(a^2 - 4b < 0)$. The general solution is $y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$, $\lambda_1 = \frac{-a}{2} + i\omega$, $\lambda_2 = \frac{-a}{2} - i\omega$, $\lambda_{1,2} = \pm i c_s k$ in our case. The exponential (increase or decay) term becomes 1 as a = 0. Therefore our k mode's solution is

Q3

$$\delta\rho_{\boldsymbol{k}} = A_{\boldsymbol{k}} e^{i\omega_k t} + B_{\boldsymbol{k}} e^{-i\omega_k t} = A_{\boldsymbol{k}} e^{i c_s k t} + B_{\boldsymbol{k}} e^{-i c_s k t} \tag{32}$$

A(k) and B(k) is the k^{th} amplitude of the wave. For ρ_{k1}... ρ_{kn} each have the solution of a linear combination of sinusoidal function with constant Amplitude.This ~~indicate~~ shows that all the fluctuations modes in static spacetime oscillate with constant amplitude, see Fig 14.

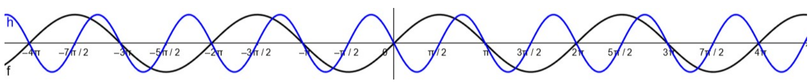


Figure 14. different mode with constant amplitude

5. Case 2: Static space with gravity

In this case, the only difference is the existance of gravity,therefore we can use the same result in case 1 in continuity Eq.24 and also the euler Eq.27 with the gravity term added. Keep in remind we only concern the 1st order perturbations

$$\partial_t \delta\boldsymbol{u} = -\frac{\nabla_{\boldsymbol{r}}\delta P}{\bar{\rho}} - \nabla_{\boldsymbol{r}}\delta\Phi \quad \text{Euler eq.} \tag{33}$$

$$\partial_t \delta\rho(\boldsymbol{r},t) = -\bar{\rho}\nabla_{\boldsymbol{r}} \cdot \delta\boldsymbol{u} \quad \text{Continuity eq.} \tag{34}$$

Using same method to the previous case, applying ∂_t the continuity ~~equ.~~ equation and $\nabla_{\boldsymbol{r}}\cdot$ the Euler equation,

$$\partial_t(\partial_t\delta\rho(\boldsymbol{r},t)) = \partial_t(-\bar{\rho}\nabla_{\boldsymbol{r}}\cdot\delta\boldsymbol{u}) \quad = \quad -[\nabla_{\boldsymbol{r}}\cdot(\bar{\rho}\partial_t\delta\boldsymbol{u}) = \nabla_{\boldsymbol{r}}\cdot(-\nabla_{\boldsymbol{r}}\delta P - \bar{\rho}\nabla_{\boldsymbol{r}}\delta\Phi)] \tag{35}$$

Combining ~~substitute~~ both equations and $\delta P = c_s^2\delta\rho$ of and ~~combine~~ using with the perturbed poisson equation $(\nabla_r^2\delta\Phi = 4\pi G\delta\rho)$ and we have

$$\partial_t(\partial_t\delta\rho(\boldsymbol{r},t)) = \nabla_{\boldsymbol{r}}\cdot(\nabla_{\boldsymbol{r}}\delta P + \bar{\rho}\nabla_{\boldsymbol{r}}\delta\Phi) \quad \rightarrow \quad (\partial_t^2 - c_s^2\nabla_r^2)\delta\rho = \bar{\rho}\nabla_r^2\delta\Phi \tag{36}$$

$$[\partial_t^2 + (c_s^2 k^2 - 4\pi G\bar{\rho})]\delta\rho = 0 \tag{37}$$

Q4 ? $K_J \equiv K_{critical}$?

our root is $i\sqrt{c_s^2k^2 - 4\pi G\bar{\rho}}$ Therefore our solution is

$$\delta\rho_{\boldsymbol{k}} = A_{\boldsymbol{k}} e^{i(\sqrt{c_s^2k^2-4\pi G\bar{\rho}})t} + B_{\boldsymbol{k}} e^{-i(\sqrt{c_s^2k^2-4\pi G\bar{\rho}})t} \tag{38}$$

Q4 that is without and This

This shows there ~~exist~~ a critical wavenumber/mode ~~which there are no~~ oscillation with constant amplitude . Critical k $= \frac{\sqrt{4\pi G\bar{\rho}}}{c_s}$ is known as Jeans wavenumber . When the mode with small wavelength->large wavenumber k > $k_{critical}$, meaning the pressure $c_s^2k^2$ dominates leading to oscillations with constant amplitude. And when, large wavelength->small wavenumber, k < kJ, gravity term dominates, the frequency ω becomes imaginary, then the solution is in a form of $Ae^{at} + Be^{-at}$ where a is real number and when time increase, the exponential decay term vanish and the first term dominate meaning the fluctuations grow exponentially. We can express the condition in Jean length which is inverse proportional to wavenumber. $\lambda = \frac{2\pi}{k_J}$

6. Case 3: Expanding Universe

6.1. Tool for Expanding space: From physical to comoving coordinates

Q5 [Important: the changes on our analyse below ~~are~~ does not convert the equations from proper to comoving coordinates but in term of comoving coordinate , it is different]

say by words

The purpose of our analysis is to study the evolution of perturbations with respect to the background FRW Universe, therefore we translate the three fluid equations from physical coordinates and in terms of full physical quantities , to comoving coordinates and in terms of perturbation quantities. In the case of expansion space, it is best to use comoving coordinates $\boldsymbol{x}$ which ~~is~~ are unchanged and the relation with proper distance is $\boldsymbol{r(t)} = a(t)\boldsymbol{x}$. taking the Derivative $\boldsymbol{r(t)}$ give:

$v_H = Hr$ due to the

$$\overbrace{\boldsymbol{u}(t)}^{\text{proper velocity}} = \boldsymbol{r}' = \dot{a}x + a\dot{x} = \dot{a}\cdot\frac{r}{a} + a\cdot\dot{x} = \overbrace{H(t)\boldsymbol{r}}^{background} + \overbrace{\boldsymbol{v}}^{perturbation} \tag{39}$$

the velocity ~~that obtain~~ hubble flow (which describes the motion of astronomical objects due solely to this expansion) + peculiar velocity (v) which refer to velocity of an object relative to a rest frame(in terms of comoving) . Meaning that it is also be interpreted in a form of u = $\bar{\boldsymbol{u}} + \delta(\boldsymbol{u})$ (background + perturbation). One ~~of the~~ example of peculiar velocity is in the physical observation of galaxy's velocity, it deviate from the Hubble flow and may recede from us. Peculiar velocity is a ~~realization~~ consequence of perturbations

We also need to convert ∇_r and $(\frac{\partial}{\partial t})_t$ as they are no longer t and r independent in an expanding case. The proper spatial derivative at fix convert to comoving derivative

... fix ... what?

$$\frac{\partial}{\partial r(t)} = \frac{\partial}{\partial a(t)x} = \boldsymbol{\nabla_r} = a^{-1}\boldsymbol{\nabla_x} \tag{40}$$

By vector transformation law from old to new coordinate system $\frac{\partial f}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial f}{\partial x'^j}$ we transform the proper time derivative in terms of comoving time derivative

$$\frac{\partial}{\partial t_r} = \frac{\partial t_x}{\partial t_r}\frac{\partial}{\partial t_x} + \frac{\partial x}{\partial t_r}\frac{\partial}{\partial x} \tag{41}$$

Since $\frac{\partial t_x}{\partial t_r} = 1$ and $\frac{\partial x}{\partial t_r} = \frac{\partial a^{-1}r}{\partial t_r} = -a^{-2}\dot{a}\boldsymbol{r} = -a^{-2}\dot{a}a\boldsymbol{x} = -H\boldsymbol{x}$, keep in mind that the time derivatives is at fixed (r) and at fixed (x) since we only consider the conversion of coordinate , meaning the object in space don't have any movement and only follow hubble flow, so we can treat $\boldsymbol{r}$ as constant in $\frac{\partial a^{-1}r}{\partial t_r}$, unlike Eq 39 .

$$\frac{\partial}{\partial t_r} = \frac{\partial}{\partial t_x} - H\boldsymbol{x}\cdot\frac{\partial}{\partial \boldsymbol{x}} = \frac{\partial}{\partial t_x} - 3H \tag{42}$$

And this transformation is same as the operator in Euler equation we derived before, the total time derivative $D_t = [\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)]$ which the first term describes the rate of change of a fix position and the second term is the rate in change in time due to a change in position, and the only difference is these derivatives is reside in comoving space regardless of hubble expansion (comoving Eulerian to a comoving Lagrangian formulation) .

6.2. Linearized Fluid Equations for Expanding Universe

As our main goals ~~is~~ are to describe the growth of small inhomogeneities in the linear regime in order to understand the formation of largescale structures in the universe. We define a term called overdensity field which is the deviation from smoothness at a given point at a given time. It is the perturbation field over the background density. Also the dependance of density pertubation δ change from $(r)$to(x). The density pertubation at comoving location x is $\delta(\boldsymbol{x},t) = \frac{\rho(x,t)-\overline{\rho(t)}}{\overline{\rho(t)}} = \frac{\delta(x,t)\overline{\rho(t)}}{\overline{\rho(t)}} = \frac{\delta\overline{\rho}}{\overline{\rho}}$. The density at comoving location x is $\rho(\boldsymbol{x},t) = \overline{\rho(t)}(1+\delta(\boldsymbol{x},t))$ where $\overline{\rho(t)}$ is the global uniform background density and also $c_s(t)$ depend on time which unlike the case of a static spacetime. Now we have the tool to treat the case of expanding universe and in the following step we will perform linearization of the Continuity equation, Euler equation, Poisson equation which only preserve zero order $\overline{\rho}$ and first order term perturbations of ρ and ν. And use the method in the previous case again, and derive the master equation.This is the quantum fluctuations in inflaton which is believe to ~~the~~ random process in the early Universe.
be generated in a stocastic

6.3. Continuity equation in Expanding Universe

Reword Substitute the tools $\boldsymbol{\nabla}_r$, ∂_t ,ν in terms of comoving space to the Continuity equation with first order pertubation.We have

$$[\frac{\partial}{\partial t_x} - H\boldsymbol{x}\cdot\boldsymbol{\nabla}_x][\overline{\rho}(1+\delta)] + a^{-1}\boldsymbol{\nabla}_x\cdot[\overline{\rho}(1+\delta)(Ha\mathbf{x}+\boldsymbol{v})] = 0 \tag{43}$$

When we obtain the zero order term (which is remain background term and drop perturbations δ and ν) of the Continuity equation:

$$[\frac{\partial}{\partial t_x} - H\boldsymbol{x}\cdot\boldsymbol{\nabla}_x][\overline{\rho}] + a^{-1}\boldsymbol{\nabla}_x\cdot[\overline{\rho}(Ha\boldsymbol{x})] = 0 \tag{44}$$

Since the background density is time-varying, but constant in space $\boldsymbol{\nabla}_x\overline{\rho} = 0$ The second term vanish.We obtain a result same as the energy conservation equation without the pressure term where we introduce in lecture 2. Since $\boldsymbol{\nabla}_x\cdot\boldsymbol{x} = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3$.

?

$$\frac{\partial\overline{\rho}}{\partial t_x} + \overline{\rho}(a^{-1}\boldsymbol{\nabla}_x)\cdot(Ha\boldsymbol{x}) \quad\rightarrow\quad \frac{\partial\overline{\rho}}{\partial t_x} + 3H\overline{\rho} = 0 \quad \text{(Zero Order)} \tag{45}$$

Solving the Eq. we obtain:

$$\frac{\dot{\overline{\rho}}}{\overline{\rho}} = -3\frac{\dot{a}}{a} \rightarrow \int\frac{\dot{\overline{\rho}}}{\overline{\rho}}\mathrm{d}t = \int -3\frac{\dot{a}}{a}\mathrm{d}t \rightarrow \ln(\overline{\rho}(t)) = -3\ln(a(t)) \rightarrow \overline{\rho}(t) = a(t)^{-3} \tag{46}$$

which gives the background homogeneous mass density $\propto a^{-3}$. Now we consider first order in fluctuations which drop (the zero order equation, products of δ and ν) and only remain(the perturbations δ and ν term) and substitute $\boldsymbol{\nabla}_x\cdot\boldsymbol{x} = 3$ give:

$$[\frac{\partial}{\partial t_x} - H\boldsymbol{x}\cdot\boldsymbol{\nabla}_x][\overline{\rho}\delta] + \overbrace{a^{-1}\boldsymbol{\nabla}_x\cdot\overline{\rho}Ha\boldsymbol{x}}^{background} + \overbrace{a^{-1}\boldsymbol{\nabla}_x\cdot\overline{\rho}\boldsymbol{v}}^{1-order} + \overbrace{a^{-1}\boldsymbol{\nabla}_x\cdot\overline{\rho}\delta Ha\boldsymbol{x}}^{1-order} + \overbrace{a^{-1}\boldsymbol{\nabla}_x\cdot\overline{\rho}\delta\boldsymbol{v}}^{2-order} = 0 \tag{47}$$

$$[\frac{\partial}{\partial t_x} - H\boldsymbol{x}\cdot\boldsymbol{\nabla}_x][\overline{\rho}\delta] + a^{-1}\boldsymbol{\nabla}_x\cdot[\overline{\rho}(Ha\boldsymbol{x}\delta+\boldsymbol{v})] = 0 \quad \text{(First Order)} \tag{48}$$

The second term vanishes since $\boldsymbol{\nabla}_x\overline{\rho} = 0$ and apply into the chainrule to $(\overline{\rho}\delta)$ gives :

$$\delta\frac{\partial\overline{\rho}}{\partial t_x} + \overline{\rho}\frac{\partial\delta}{\partial t_x} + 3H\overline{\rho}\delta + a^{-1}\overline{\rho}\boldsymbol{\nabla}_x\cdot\boldsymbol{v} = 0 \tag{49}$$

Substitute equation 45 into 49 gives:

$$\overbrace{[\frac{\partial \bar{\rho}}{\partial t_x} + 3H\bar{\rho}]}^{0}\,\delta + \bar{\rho}\frac{\partial \delta}{\partial t_x} + a^{-1}\bar{\rho}\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0 \rightarrow \bar{\rho}\frac{\partial}{\partial t_x}(\delta) + a^{-1}\bar{\rho}\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0 \tag{50}$$

$$\dot{\delta} = -a^{-1}\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v} \tag{51}$$

6.4. Euler equation in Expanding Universe Do word...

Similarily we also substitute the transformation tool and perturbation give

$$\frac{\partial \boldsymbol{v}}{\partial t_x} + \frac{\dot{a}}{a}\boldsymbol{v} + (\boldsymbol{v} \cdot \frac{\boldsymbol{\nabla}_x}{a})\boldsymbol{v} = -\frac{\boldsymbol{\nabla}_x(\overline{P} + \delta P)}{a\bar{\rho}(1+\delta)} - \frac{\boldsymbol{\nabla}_x(\overline{\Phi} + \delta\Phi)}{a} \tag{52}$$

then $\boldsymbol{\nabla}_x\overline{P} = 0$ and $\boldsymbol{\nabla}_x\overline{\Phi} = 0$,and we perform linearization and drop terms ρ^2,ν^2,$\rho\nu$,the third term vanish, the denominator drop δ when being divide by $\boldsymbol{\nabla}_x\delta P$ (2 order)

$$\frac{\partial \boldsymbol{v}}{\partial t_x} + \frac{\dot{a}}{a}\boldsymbol{v} = -\frac{\boldsymbol{\nabla}_x\delta P}{a\bar{\rho}} - \frac{\boldsymbol{\nabla}_x\delta\Phi}{a} \quad \text{(First order)} \tag{53}$$

What case? use sections numbers

6.5. Poisson equation in Expanding Universe

Same as case 2, the perturbate Poisson equation is $\nabla_r^2\delta\Phi = a^2\nabla_x^2\delta\Phi = 4\pi G a^2\bar{\rho}\delta$

6.6. Master equation in Expanding Universe

Now, we combine it to $\partial_{tr} = \frac{\partial}{\partial t_x} - H\boldsymbol{x} \cdot \frac{\partial}{\partial x}$ the Continuity equation 51 and $\boldsymbol{\nabla_r} = a^{-1}\boldsymbol{\nabla}_x$the Euler equation 53, yield our master equation. ?

6.6.1. ∂_{tr} (Continuity equation)

$$\partial_{tr}(\dot{\delta}) = -\partial_{tr}(a^{-1}\boldsymbol{\nabla}_x \cdot \boldsymbol{\nu}) \quad \rightarrow \quad (\partial_{tx} - 3H)\dot{\delta} = (\partial_{tx} - 3H)(-a^{-1}\boldsymbol{\nabla}_x \cdot \boldsymbol{\nu})) \quad \rightarrow \tag{54}$$

$$\text{chainrule as a(t)} \rightarrow \quad \ddot{\delta} - 3H\dot{\delta} = (-a^{-1}\boldsymbol{\nabla}_x \cdot \dot{v} + Ha^{-1}\boldsymbol{\nabla}_x \cdot \dot{v}) + 3Ha^{-1}\boldsymbol{\nabla}_x \cdot \dot{v} \quad \rightarrow \tag{55}$$

$$\ddot{\delta} - 3H\dot{\delta} = (\overbrace{-a^{-1}\boldsymbol{\nabla}_x \cdot \dot{v}}^{\dot{\delta}} + H\overbrace{a^{-1}\boldsymbol{\nabla}_x \cdot \dot{v})}^{-\dot{\delta}} + 3H\overbrace{a^{-1}\boldsymbol{\nabla}_x \cdot \dot{v}}^{-\dot{\delta}} \quad \rightarrow \text{substituate eq.51} \tag{56}$$

$$\ddot{\delta} + H\dot{\delta} = -a^{-1}\boldsymbol{\nabla}_x \cdot \dot{v} \tag{57}$$

6.6.2. $\nabla_r\cdot$(Eular equation)

$$-[a^{-1}\boldsymbol{\nabla}_x \cdot (\dot{\boldsymbol{v}} + \frac{\dot{a}}{a}\boldsymbol{v}) = a^{-1}\boldsymbol{\nabla}_x \cdot (-\frac{\boldsymbol{\nabla}_x\delta P}{a\bar{\rho}} - \frac{\boldsymbol{\nabla}_x\delta\Phi}{a})] \quad \text{substituate eq.57} \rightarrow \tag{58}$$

$$\ddot{\delta} + H\dot{\delta} + a^{-1}\boldsymbol{\nabla}_x \cdot H\boldsymbol{v} \quad \rightarrow \quad \ddot{\delta} + 2H\dot{\delta} = \frac{\nabla_x^2\delta P}{a^2\bar{\rho}} + \frac{\nabla_x^2\delta\Phi}{a^2} \tag{59}$$

Substitate poisson Eq. equ.6.5 yield

wrong label

$$\ddot{\delta} + 2H\dot{\delta} = \frac{\nabla_x^2\delta P}{a^2\bar{\rho}} + 4\pi G\bar{\rho}\delta \quad \text{Master equation} \tag{60}$$

Since the above equation is linear, we obtain, for each independent k mode by fourier transform, since $\nabla_x^2 \rightarrow -k^2$, $\delta P = c_s^2\bar{\rho}\delta$

$$\ddot{\delta} + 2H\dot{\delta} = \frac{\nabla_x^2 \delta P}{a^2\bar{\rho}} + 4\pi G\bar{\rho}\delta \quad \rightarrow \quad \ddot{\delta} + 2H\dot{\delta} + [\frac{k^2 c_s^2}{a^2} - 4\pi G\bar{\rho}]\delta = 0 \tag{61}$$

Now, we obtain the master equation which describes the time evolution of density fluctuations. Comparing with the master equations obtained in the absence of an expanding background, we see that the only difference is the presence of second term "Hubble damping term" which expresses how expansion suppresses perturbation growth. This term will moderate the exponential instability of the background to long wavelength density fluctuations. In addition, it will lead to a damping of the oscillating solutions on short wavelengths. The third term is refer to pressure term which expresses how pressure gradients suppress the perturbation growth. The fourth term is refer ed on the gravitational term which expresses how gravity promotes perturbation growth. In addition, there are also an extra Q7 term related ~~refer~~ to entropy, but it is out of our scope and not shown in here. For this master equation, there are two case s. Case 1: Below the Jeans length, the fluctuations oscillate with decreasing amplitude. Case 2: Above the Jeans' length, the fluctuations experience power-law growth, rather than the exponential growth we found for static space. These properties are shown in below in different epoch s. And it is important to notice that the hubble term and the density in gravity term depend on time, unlike the static cases that the coefficients are constant, this will effect the solution we obtain.

define it here

7. Dark Matter inside Hubble

Keep in ~~remind~~ mind that o ~~On~~ on scales much smaller than the Hubble radius λ_H we can employ the Newtonian theory to study Q8 perturbations in the nonrelativistic matter, also suitable to describle the evolution dark matter on sub-Hubble scales.

7.1. Matter-dominated era: dark matter fluctuations

Say what λ_H is

We now focus on the time in the matter-dominated era, the time evolution which past recombination, when the baryonic matter can be treated as a pressureless fluid ($c_s = 0$, linearised CDM fluctuations) and ignore radiation pressure term. The condition for this solution is $\lambda_J \ll \lambda \ll \lambda_H$, in fact for Non-linear effect that produce a finite, small, sound speed which give tiny pressure, but it do not effect the large wavelength perturbation.

As we know that this period $a \propto t^{2/3} \rightarrow \frac{da}{dt} \propto \frac{dt^{\frac{2}{3}}}{dt} \rightarrow \frac{\dot{a}}{a} \propto \frac{\frac{2t^{\frac{-1}{3}}}{3}}{t^{\frac{2}{3}}} \rightarrow H \propto \frac{2}{3t}$ and we

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}\delta_m = 0 \quad \text{where m refer s matter} \tag{62}$$

From Friedman equation $H^2 = \overbrace{\frac{8\pi G\rho}{3}}^{matter-density} - \overbrace{\frac{kc^2}{a^2}}^{curvature} + \overbrace{\frac{\Lambda c^2}{3}}^{dark-energy}$ and in Matter-dominated era, the first term dominate, $H^2 = \frac{8\pi G\bar{\rho}_m}{3}$. Therefore, a homogenous and isotropic perfect fluid imply $4\pi G\bar{\rho}_m = \frac{3}{2}H^2$, the equation become

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0 \tag{63}$$

Let $\delta = t^r$ and substiute to equ. yield

$$(t^r)'' + \frac{4}{3t}(t^r)' - \frac{2}{3t^2}(t^r) = 0 \rightarrow \frac{(3r^2 + r - 2)(t^{r-2})}{3} = 0 \rightarrow r = \frac{2}{3} \quad \text{or} \quad r = -1 \tag{64}$$

$$\delta(t) = C_1(k)t^{\frac{2}{3}} + C_2(k)t^{-1} \quad \text{or} \quad \delta(t) = C_1(k)a + C_2(k)a^{\frac{-3}{2}} \quad \text{pressureless (bayonic/dark matter)} \tag{65}$$

The solution give a superposition of two mode s. One is damping mode which dissapears with time, and one mode with power law growth unlike the case of static space which has ~~is~~ exponential growth. This growing mode mode will evolve with time and play a leading role in the formation of large-scale structure. The plot of the sum of two mode is in ~~refer to~~ Fig. 16 which shows the hubble expansion reduce the growth significantly when compared to. In Fig. 15 shows that the growth of perturbations when there ~~are~~ is no hubble expansion, the constant terms C_1, C_2 is set as 1 as illustration.

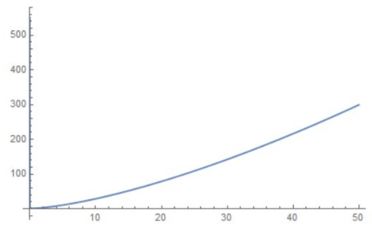


Figure 15. Without the suppression of Hubble expansion

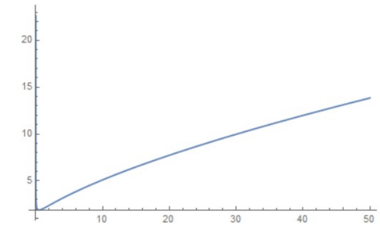


Figure 16. With the suppression of Hubble expansion

Q9

Note that baryonic perturbations cannot grow until matter has decoupled from radiation at recombination , therefore baryonic perturbations grow as scale factor $a \propto \delta_m \propto t^{\frac{2}{3}}$ after recombination (t > t rec). On the other hand, Dark matter are already collisionless and fluctuations in their density can grow immediately after equipartition.$a \propto \delta_{dm} \propto t^{\frac{2}{3}}, (t_{eq} < t < t_{rec})$. The growth of pertubation indicate the infall of matter and increase in the density of a location until the pressure of the inner core of the object is sufficient to counterbalance, then object like galaxy cluster begin to ~~is~~ form.

In subhorizon scales, the radiation perturbation δ_r behave as acoustic oscillations on both matter and radiation dominate era, the propagation of sound waves in the photon fluid give non zero pressure. Since recombination of protons and neutrons occurs in the matter era, we expect that at the time the CMB decouples ~~is emitted~~ different Fourier modes are in ~~which is~~ different phases of their oscillation. However, before the radiation fully decouples, the baryon fluid is still tightly coupled with photon , the small finite sound speed which give tiny pressure do a effect and prevent the growth of baryonic structures in small scale pertubations. So the pressure term should be keep in the Eq. 66 , and we see in the pressure term for large scale (k is small), the term is approximate to zero, but in small scale bayonic perturbation($k > k_J$) or $\lambda_J >> \lambda$, the pressure do a effect is non-negligable:

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m + (\frac{k^2 c_s^2}{a^2} - \frac{2}{3t^2})\delta_m = 0 \tag{66}$$

The solution is a combination of two bessel functions multiplied by ~~with~~ $t^{-1/6}$ which is a ~~function that are~~ decaying and oscillating function of ~~with~~ frequency $\omega = \frac{c_s k}{a}$. Later on the decay becomes smaller and settle to a "constant" amplitude. This solution and plots are done by mathematica and the values is set to an arbitrary values for illustration, ~~which is~~ $C_1 = C_2 = 1, \omega = 2$.

$$\delta_m = C_1 t^{-\frac{1}{6}} J_{\frac{5}{6}}(\frac{c_s k}{a} t) + C_2 t^{-\frac{1}{6}} J_{\frac{-5}{6}}(\frac{c_s k}{a} t) \quad \text{(bayonic matter)} \tag{67}$$

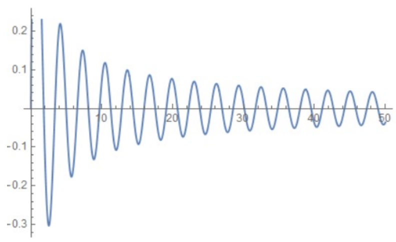


Figure 17. First term

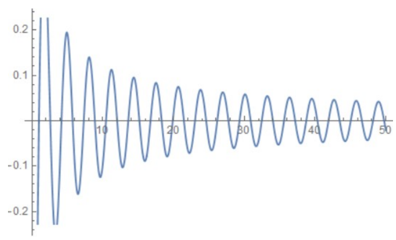


Figure 18. Second term

7.2. Radiation-dominated era: dark matter fluctuations

Consider the radiation dominated regime, the fluid is a mixture of radiation and collisionless particles. Therefore, the total density fluctuation is a combination of both $\delta_{total} = \delta_r + \delta_{dm}$ and provide the source of gravitational potential fluctuation $\delta\Phi$. On subhorizon scales radiation density fluctuation δ_r will exhibit an oscillatory behaviour (sound wave), this give the time average $<\delta_r> = 0$ for scale smaller than hubble radias.

$$\ddot{\delta}_{dm} + 2H\dot{\delta}_{dm} - 4\pi G(\overline{\rho}_{rad}\delta_{rad} + \overline{\rho}_{dm}\delta_{dm}) = 0 \rightarrow \ddot{\delta}_{dm} + 2H\dot{\delta}_{dm} - 4\pi G\overline{\rho}_{dm}\delta_{dm} = 0 \tag{68}$$

In this period, $H^2 = \frac{8\pi G(\overline{\rho}_{rad}+\overline{\rho}_{dm})}{3}$ Divide ing the equation (68) by H^2

$$\frac{\ddot{\delta}_{dm}}{H^2} + \frac{2}{H}\dot{\delta}_{dm} - \frac{3\overline{\rho}_{dm}\delta_{dm}}{2[\overline{\rho}_{dm} + \overline{\rho}_{rad}]} = 0 \tag{69}$$

If we now use that deep in the radiation dominated era (refer to early time of radiation dominated era), $\overline{\rho}_{dm} \ll \overline{\rho}_{rad}$, then the term on the RHS can be ignored. Also, since a $\propto t^{1/2} \rightarrow \frac{da}{dt} \propto \frac{dt^{\frac{1}{2}}}{dt} \rightarrow \frac{\dot{a}}{a} \propto \frac{\frac{t^{\frac{-1}{2}}}{2}}{t^{\frac{1}{2}}} \rightarrow H \propto \frac{1}{2t}$ The differential equation then simplifies to

Define all acronyms

$$\frac{\ddot{\delta}_{dm}}{H} + 2\dot{\delta}_{dm} = 0 \rightarrow \ddot{\delta}_{dm} + \frac{\dot{\delta}_{dm}}{t} = 0 \tag{70}$$

The solution is

$$\delta_{dm} = C_1(k) + C_2(k)\ln t \quad \text{or} \quad C_1(k) + C_2(k)\ln a \quad \text{(dark matter)} \tag{71}$$

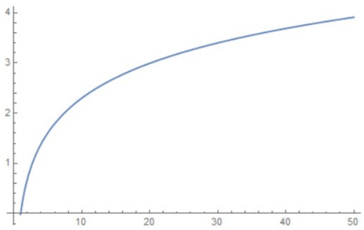


Figure 19. ln t (improve caption)

This solution also indicate the perturbations have a much slower growth on subhorizon scales in the radiation era, as we compare figs. 16 to 19. The stagnation of growth in pressureless matter perturbations during radiation dominated era is known as the Meszaros effect. The Meszaros effect is simply a manifestation of the fact that the Hubble drag term during the radiation dominated era is larger than during the matter dominated era. The dark matter perturbations can only grow significantly when the background gravitational potential is sufficiently strong to trigger their collapse. This happan to δ_{dm} in superhorizon scales.

7.3. plotting of dark matter fluctuations during Radiation and Matter domination Era

Q10

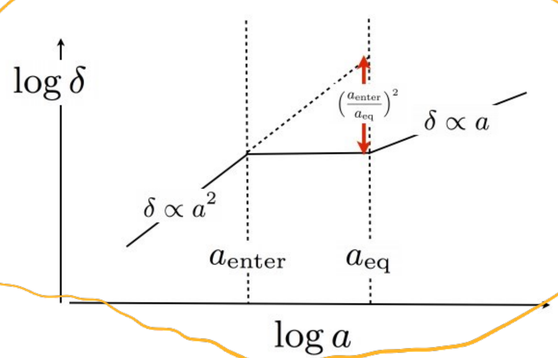


Figure 20. log δ_m vs log a plot \cite source of the figure

The fig. 20 show the time evolution of a perturbation $\delta(k)$ with size $L = \frac{2\pi}{k}$ enter the horizon during the radiation dominated era at scale factor a_{enter}. The time evolution goes through three different phases:

(i) $\delta \propto a^2$ before horizon entry which follows from general relativistic perturbation theory which is out of our scope.

give cite of source

(ii) When redshifts z > z_{eq} or a < a_{eq} , where z_{eq} $\sim$ 24000 is moment Matter-radiation equality happens . The Universal energy density is dominated by radiation.Althrough,in radiation domination $H \propto \frac{1}{2t}$,which expand slower than matter dominance epoch($H \propto \frac{2}{3t}$), meaning the suppression due to expansion of universe is relatively lower, but also combine with the fact that the dark matter density is so small, leading to the gravity term which play the rule to enhance perturabation is now neglected. These factor lead to the freeze out of growth δ = constant after the pertubation enter the horizon.This Meszaros effect suppress the growth by a factor of $\frac{a_{enter}}{a_{eq}}$.

(iii)When a > a_{eq}, the matter density starts to dominate the Universal energy density, the dark matter perturbation grows as $\delta \propto a$.

7.4. Dark Energy Λ dominated era: Bayonic/dark matter fluctuations

In the dark energy dominated era , the universe begins expanding in an accelerated way ,and mainly compose of dark energy and also minority of bayonic matter ,therefore the total density fluctuation is suppose to contribute by both dark energy + matter ,however due to the reason that since dark energy does not have any local gravitational effects, but rather a global effect on the universe as a whole,the term δ_Λ is drop

$$\ddot{\delta}_{dm} + 2H\dot{\delta}_{dm} - 4\pi G\bar{\rho}(\delta_{dm} + \overbrace{\delta_\Lambda}^{0}) = 0 \tag{72}$$

Also, the $\bar{\rho}_m \propto a^{-3}$ become very small in this period, beacause a increase exponentially by time $a \propto e^{\sqrt{\Lambda}t}$ where Λ is constant,in freidmann equation $H^2 \approx \frac{\sqrt{\Lambda c^2}}{3} = constant \gg 4\pi G\bar{\rho}_{dm}$. So the last term can be ignore

This give a solution of

$$\ddot{\delta}_{dm} + 2H\dot{\delta}_{dm} \approx 0 \rightarrow \delta_{dm} = C_1(k) + C_2(k)e^{-2Ht} = C_1(k) + C_2(k)a^{-2} \tag{73}$$

with a constant mode and a rapidly decaying mode. Hence, the growth of perturbations stops when the cosmological constant takes over, this indicate there was sufficient time for large scale structures to form in the universe before dark energy domination.

8. Summary

We have demonstrated the idea of pertubation theory relate to the structure formation ,and shown the pertubation through the observation in CMB. We have studied the concept of three fluid equation which successfully describle the perturbation fields on scales well inside the Hubble radius. Fourier transform is introduced to decompose the perterbation in to the sum of all scale inside Hubble radius, which simply the problem from P.D.E to O.D.E, and obtain the master equation .We have seperated 3 cases to study the behavior of the perturbation in 1) static universe without gravity , 2) static universe with gravity , 3) Expanding universe with gravity, and have shown the solution which describle the dynamic of the perturbation . In case one , the flucturation behave as oscillation with constant amplitude. In case two, for small scale perturbation oscillate with constant amplitude and for large scale, fluctuations grow exponentially. In case three, we desired to understand the dynamic of perturbation respect to the background, so we have introduced the comoving tools and expression the three fluid equation in term of comoving coordinate. And then we analyzed the solution of dark matter pertubation in three different era. 1)Matter dominate era 2)Radiation dominate era 3) Dark energy dominate era. For case 1, the fluctuations of dark matter grow proportional to scale factor a immediately after equipartition , and after recombination for bayonic matter, and the hubble expansion play the rule on this suppression. Also , for small scale bayonic matter perturbation , the pressure term also effect as a result in a ocillation with decay behavior. In case 2, the dark matter perturbations have a much slower growth *lna* due to Meszaros effect. For case 3, the growth of perturbations stop and start at a constant value, and universe start expanding in acceleralation and the cosmological object evolve.

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9. References

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