

Complementary Quantum Mechanics

Homework Set 2: Perturbation Theory

This set covers stationary perturbation theory, sudden changes of the Hamiltonian, and time-dependent transitions. Show the essential steps and state the approximation used.

Useful notation and relations

For a non-degenerate eigenstate $|n\rangle$ of $\hat{H}_0$, the first two energy corrections are

$$E_n^{(1)} = \langle n | \hat{V} | n \rangle, \quad E_n^{(2)} = \sum_{k \neq n} \frac{|\langle k | \hat{V} | n \rangle|^2}{E_n^{(0)} - E_k^{(0)}}.$$

For a perturbation $\hat{V}(t)$ acting during $0 \leq t \leq \tau$, the first-order transition amplitude is

$$C_f^{(1)}(\tau) = -\frac{i}{\hbar} \int_0^\tau \langle f | \hat{V}(t) | i \rangle e^{i\omega_{fi}t} dt, \quad \omega_{fi} = \frac{E_f^{(0)} - E_i^{(0)}}{\hbar}, \quad f \neq i.$$

For a sudden Hamiltonian change, the state is unchanged during the switch. The final-state probabilities are

$$P_n = |\langle n; f | \psi_i \rangle|^2.$$

A. Stationary perturbation theory

Problem 1. Two-state Hamiltonian: exact and perturbative energies

Consider a two-dimensional Hilbert space with Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, represented in an orthonormal basis by

$$H_0 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}.$$

The matrix entries are expressed in the same energy unit, and λ is a real parameter.

- Find the eigenvalues and normalised eigenvectors of H_0 .
- Find the *exact* eigenvalues of H as functions of λ .
- Calculate the first- and second-order corrections to both energy levels using non-degenerate stationary perturbation theory.
- Expand the exact eigenvalues for small λ , compare with part (c), and identify the dimensionless parameter controlling the perturbative approximation.

Problem 2. Harmonic oscillator in a uniform electric field

A particle of mass m and signed charge q moves along the x -axis in a harmonic potential of angular frequency ω . A uniform static electric field of magnitude $\mathcal{E}$ points along the positive x -axis. The Hamiltonian is

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 - q\mathcal{E}x.$$

- Complete the square in the potential energy and determine the exact energy eigenvalues. Find the displacement of the oscillator's equilibrium position.

- (b) Take $\hat{V} = -q\mathcal{E}x$ as the perturbation. Show that the first-order energy correction vanishes for every oscillator level.
- (c) Calculate the second-order energy correction using harmonic-oscillator matrix elements. Compare it with the exact result from part (a).

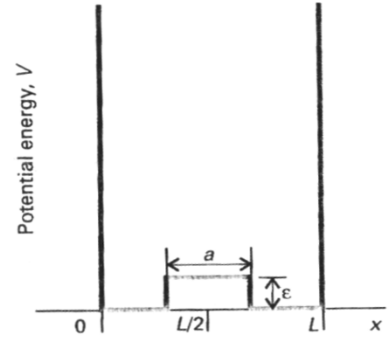
Problem 3. A local perturbation in an infinite square well

A particle of mass m is confined to the interval $0 < x < L$ by infinitely high walls. The unperturbed eigenfunctions and energies are

$$\psi_n^{(0)}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n^{(0)} = \frac{\pi^2 \hbar^2 n^2}{2mL^2}, \quad n = 1, 2, \dots$$

A small, rectangular perturbation of height $\varepsilon > 0$ and width a is placed symmetrically about the centre of the well, with $0 < a < L$:

$$V(x) = \begin{cases} \varepsilon, & (L-a)/2 \leq x \leq (L+a)/2, \\ 0, & \text{otherwise inside the well.} \end{cases}$$



- (a) Write the integral giving the first-order energy correction $E_n^{(1)}$ for the state with quantum number n .
- (b) Evaluate the integral for arbitrary $n \geq 1$ and express the energy through first order in ε .
- (c) Examine how the correction depends on n and on the width a . Check your answer in the limits $a \rightarrow 0$ and $a \rightarrow L$.

B. Sudden approximation

Problem 4. Sudden application of an electric field

A particle of mass m and signed charge q is initially in the ground state of a one-dimensional harmonic oscillator with angular frequency ω and no applied electric field. At $t = 0$, a uniform electric field $\mathcal{E}$ along the positive x -axis is switched on suddenly and then held constant. The Hamiltonian after the switch is the one used in Problem 2.

- (a) Find the ground-state wavefunction before the switch and identify the stationary eigenstates of the Hamiltonian after the switch.
- (b) Using the sudden approximation, express the probability P_n of measuring the *new* energy level with quantum number n immediately after the switch as an overlap integral. Evaluate this probability for arbitrary $n = 0, 1, 2, \dots$
- (c) Find the probability that the system occupies any excited eigenstate of the new Hamiltonian. Explain why this probability is non-zero even though the wavefunction does not change during the sudden switch.

C. Time-dependent perturbation theory

Problem 5. Two-state system under a constant pulse

A two-state system has orthonormal eigenstates $|1\rangle$ and $|2\rangle$ of $\hat{H}_0$, with energies $E_1 < E_2$. Define $\omega_{21} = (E_2 - E_1)/\hbar$. At $t = 0$ the system is in $|1\rangle$. A weak, time-independent Hermitian perturbation $\hat{V}$ acts only during $0 \leq t \leq \tau$; it vanishes before and after that interval.

- (a) Write the complete 2×2 Hamiltonian in the basis of the unperturbed states $|1\rangle$ and $|2\rangle$, and determine its eigenvalues.

- (b) Calculate exactly the transition probability $P_{1 \rightarrow 2}(\tau)$ and determine the survival probability $P_{1 \rightarrow 1}(\tau)$ using normalization.
- (c) Obtain the weak-perturbation limit of the exact transition probability and compare it with the result of first-order time-dependent perturbation theory. Identify the conditions for agreement.

Problem 6. Two-state system driven near resonance

Consider the same two-state system as in Problem 5, initially in $|1\rangle$. A weak time-dependent perturbation is applied during the interval $0 \leq t \leq \tau$:

$$\hat{V}(t) = \hat{V} \cos(\Omega t),$$

where $\hat{V}$ is a time-independent Hermitian operator, $\Omega > 0$ is the driving frequency, and $\omega_{21} = (E_2 - E_1)/\hbar$ is the atomic transition frequency. The perturbation vanishes outside the specified interval.

- (a) Calculate the first-order transition amplitude $C_2^{(1)}(\tau)$, retaining both terms with frequencies $\omega_{21} - \Omega$ and $\omega_{21} + \Omega$.
- (b) Consider excitation near resonance, $\Omega \simeq \omega_{21}$. Assuming $|\Omega - \omega_{21}|\tau \lesssim 1$ and $\omega_{21}\tau \gg 1$, show that the contribution with frequency $\omega_{21} + \Omega$ can be neglected. Obtain the transition probability $P_{1 \rightarrow 2}(\tau)$ within this approximation.
- (c) Calculate the transition probability at exact resonance, $\Omega = \omega_{21}$. Determine the condition under which first-order time-dependent perturbation theory remains valid.

Problem 7. Periodic perturbation and resonant excitation

A system has non-degenerate unperturbed eigenstates $|\psi_k\rangle$, with energies E_k . Initially, at $t = 0$, it is in the ground state $|\psi_0\rangle$. A weak periodic perturbation is applied for $0 \leq t \leq \tau$:

$$\hat{V}(t) = \hat{A}e^{-i\Omega t} + \hat{A}^\dagger e^{i\Omega t}, \quad \Omega > 0.$$

The operator $\hat{A}$ is time independent, and $\hat{V}(t) = 0$ outside the stated interval. Define the transition frequency $\omega_{k0} = (E_k - E_0)/\hbar > 0$ for $k \neq 0$.

- (a) Calculate the first-order amplitude $C_k^{(1)}(\tau)$ for a transition to an excited state $|\psi_k\rangle$. Keep the contributions proportional to $\langle \psi_k | \hat{A} | \psi_0 \rangle$ and $\langle \psi_k | \hat{A}^\dagger | \psi_0 \rangle$ separate.
- (b) Obtain the leading non-zero transition probability. Identify the term responsible for resonant absorption when $\Omega \simeq \omega_{k0}$.
- (c) Determine how the position and characteristic width of the absorption peak depend on ω_{k0} and the pulse duration τ . Explain what happens at exact resonance and state the condition for first-order perturbation theory to remain applicable.